IE675 Game Theory

Lecture Note Set 4

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4 UTILITY THEORY

4.1 Introduction

This section is really independent of the field of game theory, and it introduces concepts that pervade a variety of academic fields. It addresses the issue of quantifying the seemingly nonquantifiable. These include attributes such as quality of life and aesthetics. Much of this discussion has been borrowed from Keeney and Raiffa [1]. Other important references include Luce and Raiffa [2], Savage [4], and von Neumann and Morgenstern [5].

The basic problem of assessing value can be posed as follows: A decision maker must choose among several alternatives, say $W_1, W_2, \ldots, W_n$, where each will result in a consequence discernible in terms of a *single* attribute, say X. The decision maker does not know with certainty which consequence will result from each of the variety of alternatives. We would like to be able to quantify (in some way) our preferences for each alternative.

The literature on utility theory is extensive, both theoretical and experimental. It has been the subject of significant criticism and refinement. We will only present the fundamental ideas here.

4.2 The basic theory

Definition 4.1. *Given any two outcomes A and B we write $A \succ B$ if A is preferable to B. We will write $A \simeq B$ if $A \not\succ B$ and $B \not\succ A$.*

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4.2.1 Axioms

The relations $\succ$ and $\simeq$ must satisfy the following axioms:

1. Given any two outcomes A and B, exactly one of the following must hold:
   (a) $A \succ B$
   (b) $B \succ A$
   (c) $A \simeq B$
2. $A \simeq A$ for all A
3. $A \simeq B$ implies $B \simeq A$
4. $A \simeq B$ and $B \simeq C$ implies $A \simeq C$
5. $A \succ B$ and $B \succ C$ implies $A \succ C$
6. $A \succ B$ and $B \simeq C$ implies $A \succ C$
7. $A \simeq B$ and $B \succ C$ implies $A \succ C$

4.2.2 What results from the axioms

The axioms provide that $\simeq$ is an *equivalence relation* and $\succ$ produces a *weak partial ordering* of the outcomes.

Now assume that

$$A_1 \prec A_2 \prec \cdots \prec A_n$$

Suppose that the decision maker is indifferent to the following two possibilities:

$$\textbf{Certainty option:} \quad \text{Receive } A_i \text{ with probability } 1$$

$$\textbf{Risky option:} \quad \begin{cases} \text{Receive } A_n \text{ with probability } \pi_i \\ \text{Receive } A_1 \text{ with probability } (1 - \pi_i) \end{cases}$$

If the decision maker is consistent, then $\pi_n = 1$ and $\pi_1 = 0$, and furthermore

$$\pi_1 < \pi_2 < \cdots < \pi_n$$

Hence, the π's provide a numerical ranking for the A's.

Suppose that the decision maker is asked express his preference for probability distributions over the A_i. That is, consider mixtures, p' and p'', of the A_i where

$$p'_i \geq 0 \qquad \sum_{i=1}^n p'_i = 1$$
$$p''_i \geq 0 \qquad \sum_{i=1}^n p''_i = 1$$

Using the π's, we can consider the question of which is better, p' or p'', by computing the following "scores":

$$\begin{aligned} \overline{\pi}' &= \sum_{i=1}^n p'_i \pi_i \\ \overline{\pi}'' &= \sum_{i=1}^n p''_i \pi_i \end{aligned}$$

We claim that the choice of p' versus p'' should be based on the relative magnitudes of $\overline{\pi}'$ and $\overline{\pi}''$.

Note 4.1. Suppose we have two outcomes A and B with the probability of getting each equal to p and $(1-p)$, respectively. Denote the *lottery* between A and B by

$$Ap \oplus B(1-p)$$

Note that this is not expected value, since A and B are not real numbers.

Suppose we choose p'. This implies that we obtain A_i with probability p'_i and this is indifferent to obtaining

$$A_n \pi_i \oplus A_1(1-\pi_i)$$

with probability p'_i. Now, sum over all i and consider the quantities

$$\begin{aligned} & A_n \sum_{i=1}^n \pi_i p'_i \oplus A_1 \sum_{i=1}^n (1-\pi_i) p'_i \\ \simeq \quad & A_n \sum_{i=1}^n \pi_i p'_i \oplus A_1 \left(1 - \sum_{i=1}^n \pi_i p'_i\right) \\ \simeq \quad & A_n \overline{\pi}' \oplus A_1(1-\overline{\pi}') \end{aligned}$$

So if $\overline{\pi}' > \overline{\pi}''$ then

$$A_n \overline{\pi}' \oplus A_1(1-\overline{\pi}') \succ A_n \overline{\pi}'' \oplus A_1(1-\overline{\pi}'')$$

This leads directly to the following. . .

Theorem 4.1. *If* $A \succ C \succ B$ *and*

$$pA \oplus (1-p)B \simeq C$$

then $0 < p < 1$ *and* p *is unique.*

Proof: See Owen [3]. ■

Theorem 4.2. *There exists a real-valued function* $u(\cdot)$ *such that*

1. (Monotonicity) $u(A) > u(B)$ *if and only if* $A \succ B$.
2. (Consistency) $u(pA \oplus (1-p)B) = pu(A) + (1-p)u(B)$
3. *the function* $u(\cdot)$ *is unique up to a linear transformation. In other words, if* u *and* v *are utility functions for the same outcomes then* $v(A) = \alpha u(A) + \beta$ *for some* α *and* β.

Proof: See Owen [3]. ■

Consider a lottery (L) which yields outcomes $\{A_i\}_{i=1}^n$ with probabilities $\{p_i\}_{i=1}^n$. Then let

$$\tilde{A} = A_1 p_1 \oplus A_2 p_2 \oplus \cdots \oplus A_n p_n$$

Because of the properties of utility functions, we have

$$E[(u(\tilde{A})] = \sum_{i=1}^{n} p_i u(A_i)$$

Consider

$$u^{-1}\left(E[(u(\tilde{A})]\right)$$

This is an *outcome* that represents lottery (L)

Suppose we have two utility functions u_1 and u_2 with the property that

$$u_1^{-1}\left(E[(u_1(\tilde{A})]\right) \simeq u_2^{-1}\left(E[(u_2(\tilde{A})]\right) \quad \forall\, \tilde{A}$$

Then u^1 and u^2 will imply the same preference rankings for any outcomes. If this is true, we write $u^1 \sim u^2$. Note that some texts (such as [1]) say that u_1 and u_2 are *strategically equivalent.* We won't use that definition, here, because this term has been used for another property of strategic games.

4.3 Certainty equivalents

Definition 4.2. *A* **certainty equivalent** *of lottery (L) is an outcome $\hat{A}$ such that the decision maker is indifferent between (L) and the certain outcome $\hat{A}$.*

In other words, if $\hat{A}$ is a certainty equivalent of (L)

$$\begin{aligned} u(\hat{A}) &= E[u(\tilde{A})] \\ \hat{A} &\simeq u^{-1}\left(E[u(\tilde{A})]\right) \end{aligned}$$

You will also see the terms *cash equivalent* and *lottery selling price* in the literature.

Example 4.1. Suppose outcomes are measured in terms of real numbers, say $A = x$. For any a and $b > 0$

$$u(x) = a + bx \sim x$$

Suppose the decision maker has a lottery described by the probability density $f(x)$ then

$$E[\tilde{x}] = \int x f(x)\, dx$$

Note that

$$u(\hat{x}) = E[u(\tilde{x})] = E[a + b\tilde{x}] = a + bE[\tilde{x}]$$

Taking u^{-1} of both sides shows that $\hat{x} = E[\tilde{x}]$.

Hence, if the utility function is linear, the certainty equivalent for any lottery is the expected consequence of that lottery.

Question 4.1. Suppose $u(x) = a - be^{-cx} \sim -e^{-cx}$ where $b > 0$. Suppose the decision maker is considering a 50-50 lottery yielding either x_1 or x_2. So

$$E[\tilde{x}] = \frac{x_1 + x_2}{2}$$

Find the solution to $u(\hat{x}) = E[u(\tilde{x})]$ to obtaining the certainty equivalent for this lottery. In other words, solve

$$-e^{-c\hat{x}} = \frac{-(e^{-cx_1} + e^{-cx_2})}{2}$$

Question 4.2. This is a continuation of Question 4.1. If $u(x) = -e^{-cx}$ and $\hat{x}$ is the certainty equivalent for the lottery $\tilde{x}$, show that $\hat{x} + x_0$ is the certainty equivalent for the lottery $\widetilde{x + x_0}$.

4.4 BIBLIOGRAPHY

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