Stability of a Crime Level Equilibrium

by

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October 2002

Abstract – In this paper we formulate a mathematical model of rational criminal behavior. We model the crime level in an economic market with a supply of potential criminals who differ in their opportunity cost for committing crime, reflecting differences in the value of foregone opportunities such as performing productive labor. The realized demand is influenced by the expected value for crime that depends on several socio-economic variables including wealth, police enforcement and police arrest ability. Finding the crime level equilibria suggested by this static model, we propose a dynamical systems model in which criminals periodically respond to the crime level and its associated value, bringing fluctuations in the level of crime. We study the stability of this dynamical system, finding two critical enforcement levels. Exceeding the first enforcement threshold will push the crime to stabilize (converge) to an equilibrium. Additional enforcement beyond a higher enforcement threshold will collapse the crime market to zero. Failing to meet these critical levels with a relatively small but constant enforcement pressure may lead to fluctuation in the crime market over time.
1. Introduction

Crime has been seen as a serious problem in American cities; consequently, a number of studies and programs have emerged to understand and counteract it. Criminology has been seen as a complex mix of economic, social and psychological factors that are difficult to ascertain, insignificant to alert, and highly dependent on individuals. Thus, researchers have been studying criminal activities by analyzing criminal psychology, the social environment, and economic circumstances.

Earnest work on the economic analysis of criminal behavior began in the 1960’s, culminating in an extensive economic analysis of crime and punishment by Becker (1968). Major studies by Ehrlich (1973) and Sjoquist (1973) soon followed. An extensive amount of work in the mathematical and statistical modeling of crime has transpired over the past three decades since these groundbreaking efforts. Commendable recent surveys in the literature of operations research and management science can be found in Barnett, Caulkins and Maltz (2001) and Blumstein (2002).

In an economic view, criminals are assumed to seek benefit from their criminal behavior. They decide whether or not to commit crime based on the characteristics of particular offenses, in particular, their opportunities, costs, and profits. This behavior must be understood in order to use law enforcement resources most efficiently.

In order to do so, we build on the work of Freeman, Grogger and Sonstelie (1996). That work used a fairly typical economic model of crime reward versus risk. They postulate that the expected value for crime that is a product of the return for a successful crime and the probability of success, i.e., not being arrested. What is interesting in this model is that the return and probability of arrest depends on the crime level, i.e., the payoff to a potential criminal is the
marginal return that he or she can acquire in a crime market already depleted by a current set of criminals or crimes. This provides a marginal expected return on crime, which is balanced by a reservation wage that can be had for not committing crime. This model reveals two resulting crime equilibria, which the authors examine qualitatively, both in a static and dynamic sense. The focus of our paper is to extend this model to include a reservation wage function that also depends on the crime level. We then examine the resulting equilibria both quantitatively and qualitatively in both a static and dynamic model of criminal behavior.

This paper is organized as follows. In Section 2, we formulate an explicit mathematical expected return function for crime based on the model of Freeman et al. (1996) that demonstrates the relationship between the number of crimes and some socio-economic variables including wealth, police enforcement and police arrest ability. In Section 3, we assume criminals have heterogeneous opportunity costs for crime, reflecting differences in the value of foregone opportunities such as performing productive labor among potential criminals. This differs from using a so-called reservation wage common to all criminals as assumed in Caulkins (1993), Baveja et al. (1993) and Freeman et al. (1996). We again establish an explicit mathematical form and use this to solve for the crime equilibria that equate the expected value for crime with the opportunity cost of committing crime. In Section 4, we propose a dynamical systems model in which criminals periodically respond to the crime level and its associated value bringing fluctuations in the level of crime. In Section 5, we study the stability of this dynamical system, examining the range of enforcement that can push the crime to stabilize (converge) to an equilibrium, or better, collapse to zero. Section 6 summarizes the conclusions of this paper and Section 7 offers suggestions for future work.
2. Expected Crime Value Function

In this section, we develop a general mathematical model of the expected value of committing a crime that incorporates important socio-economic variables including the crime level, wealth, the amount of police enforcement and the probability of arrest. The interactions among the variables are analyzed and the plausibility of the model is verified by examining its mathematical characteristics.

In our model, a criminal decides whether or not to commit a crime based on a rational choice that weighs the return from a successful crime against the risk of being arrested. The police investigate each crime incident and attempt to arrest the guilty party. Let the amount of law enforcement that a neighborhood receives be denoted $E$, which can be measured by the patrol hours or police budget applied in the neighborhood. The more enforcement they are able to devote to a crime incident, the more likely they are to make the arrest. Therefore, under a fixed level of enforcement, a crime has a lesser possibility of arrest in an area with a larger amount of crime incidents due to the low average enforcement devoted to it. Greenwood, Chaiken and Petersilia (1977) appear to be the first to document this inverse relationship between the number of crimes and the probability of arrest. Caulkins (1993) studies the behavior of a drug market under focused enforcement and suggests drug dealers' risk from crackdown enforcement is proportional to the total enforcement per dealer.

Following these suggestions, we assume that the probability of arrest, $P_A$, is an increasing, exponential function of the enforcement per crime given by

$$P_A(E/n) = 1 - \exp(-\alpha(E/n)), \quad (1)$$
where \( n \) is the number of crime incidents and \( \alpha \) is a positive constant. Note that, following Greenwood et al. (1977), under a fixed level of enforcement, \( E \), the probability of arrest decreases as the number of crimes increases due to the low average enforcement devoted to it.

This arrest probability function implies that the random enforcement necessary to make a particular arrest, given by random variable \( X \), is less than the available (per-incident) enforcement, \( E/n \), with probability \( P_A(X < E/n) = P_A(E/n) \). Hence \( X \) is exponentially distributed with parameter \( \alpha \). This means enforcement, \( X \), is memoryless, i.e., the amount of additional investigation needed to make a particular arrest is the same as when no investigation has been made. In this setting, the constant hazard rate implies that an arrest is equally likely in the next instant of investigation regardless of the amount of prior investigation. An exponential model has been postulated for random visual search tasks (Morawski, Drury and Karwan, 1980; Koopman, 1986) and seems appropriate here for criminal search as well.

The arrest parameter \( \alpha \) reflects the effectiveness of the per-incident enforcement \( E/n \) in making an arrest. This parameter may vary by type of crime and from neighborhood to neighborhood. Some researchers (e.g., Caulkins, Larson and Rich, 1993) have discussed the fact that the neighborhood geography and environment may play a key role in enforcement effectiveness. For instance, a community with open, well-lit areas will have a higher \( \alpha \) than a dark neighborhood with more places for criminals to hide. Similarly, an affluent community, where criminals would be more notably out of place because of their appearance or behavior, would likely have a higher \( \alpha \) than a poorer community.

Our arrest probability function implies that the crime rate increases with the number of incidents (because of the decreasing arrest rate). However, the total monetary return to a crime in a neighborhood must be limited by the wealth of the neighborhood; the more crime incidents in
the neighborhood, the less wealth that remains for other criminals. Thus, the marginal monetary return for a successful crime incident, \( r(n) \), decreases with the number of crime incidents via an appropriate positive constant, \( q \), and is proportional to a function the monetary wealth of the neighborhood, \( m \), which can be taken as the median or mean of household incomes. The marginal monetary return for the \( n \)-th successful crime is given by

\[
r(n) = c(m) \exp(-qn),
\]

where \( c(m) \) is a proportionality constant that depends on the neighborhood’s monetary wealth. Although an exponential function was chosen primarily for mathematical convenience, this function does fit the general conditions imposed by Freeman, Grogger and Sonstelie (1996). The parameters \( q \) and \( c(m) \) can be chosen to influence the shape of this function over the practical range of \( n \), so that even a linear return can be approximated using very small \( q \).

The parameters \( q \) and \( c(m) \) depend on the type of crime. A crime that requires higher level of skill to commit, and hence commands a high return, should have higher \( c(m) \). Also, crimes that are more peer-competitive (more sensitive to the number of criminals) have a higher \( q \) value. Since our focus is on a single type of crime, the values of \( c(m) \) and \( q \) are assumed to be constant.

If a criminal successfully commits the \( n \)-th crime in a neighborhood, he acquires reward \( r(n) \); if arrested, he forfeits this take. Therefore, following the general model of Freeman et al. (1996), the expected monetary return from committing a crime in a neighborhood is the product of the probability of not being arrested and the reward, viz.

\[
v(n) = r(n)*(1-P_A(E/n)) = c(m) \exp(-\alpha E/n - qn).\]

(3)
Note that \( c(m) \) is an upper bound for the expected return. A criminal can expect to gain the amount of return \( c(m) \) when no enforcement is applied and no other criminals are competing with him in the neighborhood.

The expected crime value function \( v(n) \) is positive and approaches zero as the number of crimes \( n \) approaches either zero or infinity. Furthermore, \( v(n) \) is unimodal, with a unique maximum value \( v^* = c(m) \exp(-2\sqrt{aqE}) \) under the level of enforcement \( E \) occurring when the number of crime incidents is \( n^* = \sqrt{aE/q} \), a so-called organized crime equilibrium level. Thus, \( v(n) \) increases from 0 to \( v^* \) as \( n \) increases from 0 to \( n^* \) and then decreases back to 0 as \( n \) grows large. This is due to the fact that for a small crime level \( n \) the average enforcement per incident is large and hence the arrest probability is high. In this situation, more crime incidents can reduce the arrest probability and thereby increase the expected return. On the other hand, the expected return eventually falls as the number of crimes increases past \( n^* \) due to the wealth limitations of the neighborhood.

We note that \( v(n) \) and therefore \( v^* \) decreases with an increase in enforcement pressure \( E \). This is because high enforcement increases the probability of arrest and reduces the expected return \( v(n) \) (and its maximum) that criminals receive. In contrast, the organized crime level \( n^* \) increases with an increase in enforcement since criminals prefer more crimes to share the enforcement pressure. This reduces the average enforcement per crime, thereby decreasing the possibility of their being arrested. Hence, \( n^* \) is larger in an environment of higher enforcement.
3. Crime Level Equilibrium

Studies show that criminals are greatly advised by peers and role models and well informed about actual risks (e.g., Balvig 1990; Warr 1980; Young 1988). Therefore, we assume potential criminals are rational decision makers. Based on available information, they assess the costs of committing a crime and only do so when the expected marginal monetary return exceeds their opportunity cost for doing so in terms of foregone productive wages.

In Freeman et al. (1996), the criminals' opportunity cost for crime, the so-called reservation wage (cf. Caulkins, 1993), is a constant $w$. When the expected return for committing a crime in the neighborhood exceeds this amount, the neighborhood attracts criminals. Otherwise, it is unattractive. Criminal activity is in equilibrium when the opportunity cost and the expected return for a crime are equal. Depending on the initial number of crimes when the law enforcement is initially applied, either the number of crime incidents in the neighborhood will eventually reach a stable equilibrium level or the criminal activity will collapse to zero in the neighborhood.

We generalize this notion to the case where the opportunity costs of committing a crime are heterogeneous among criminals. In this case the marginal opportunity cost, rather than a constant function $w$, is an increasing function of the crime level $n$, reflecting the fact that at that level the marginal reward to entice an additional criminal to commit a crime must be relatively large, otherwise the criminal would have already entered the crime “market”. When $n$ is very large most criminals are in the market, and only a few potential criminals with very high reservation wages are not. Put another way, as suggested by Freeman et al. (1996), when the number of criminals is large the number of workers is low, raising the reservation wage for labor.
Hence, the reservation wage is a decreasing function of the number of workers and an increasing function of the number of criminals.

We define the opportunity cost or reservation wage as a function of number of crimes via

\[ w(n) = d(m) \exp(pn), \]

where \( d(m) \) is an increasing function of the monetary wealth \( m \), and \( p \) is a positive constant. Parameters \( p \) and \( d(m) \) reflect the sensitivity of criminals to the amount of the monetary return. Although not required for all of our results, we assume that \( d(m) \) is increasing in \( m \) since, in a more affluent society, the threshold that turns workers to crime is higher. To the extent that the criminals of specific types are sensitive or insensitive to the amount of the monetary return, the criminals' opportunity cost curve will start higher or lower and climb more or less steeply. We note that the ratio of the reservation wage \( w(n) \) to the return \( r(n) \) is given by \( (d(m)/c(m)) \exp(\beta n) \) for \( \beta = p+q > 0 \), which is increasing in \( n \) as suggested by Freeman et al. (1996).

The crime level reaches equilibrium at the point(s) of intersection of the expected crime value (demand) curve and the reservation wage (supply) curve as depicted in Figure 1. That is, the number of crime incidents in equilibrium should satisfy \( v(n) = w(n) \). Two cases result in no intersections between the two curves, signaling the collapse of criminal activities. One case occurs when the minimum opportunity cost, \( w(0) = d(m) \), is greater than the upper bound of the expected return curve, \( c(m) \); the other case occurs when a large police “crackdown” pressure, \( E > E_c \), is applied, where \( E_c = k/4 \alpha \beta \), \( k = \lfloor \ln(c(m)/d(m)) \rfloor^2 \) and again \( \beta = p+q \). In these two cases, \( v(n) = w(n) \) does not have a solution since the expected return never reaches the opportunity cost of a crime. Hence the criminal activities eventually collapse. Other than these two cases, solving \( v(n) = w(n) \) gives two equilibrium solutions for the crime level \( n \) given by

\[ \hat{n} := (\sqrt{k} - \sqrt{k - 4\alpha \beta E}) / 2 \beta, \]

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\[ \hat{n} := \frac{\sqrt{k} + \sqrt{k - 4\alpha \beta E}}{2\beta}. \] (6)

Potential criminals with reservation wages greater than the maximum expected return \( v^* = v(n^*) \) will not commit crimes under enforcement level \( E \). Thus, solving

\[ w(n) = d(m) \exp(pn) = c(m) \exp(-2\sqrt{\alpha qE}) = v(n^*), \]

for \( n \), we have an upper bound on the crime level given by

\[ n_{\max} := \frac{\sqrt{k} - 2\sqrt{\alpha qE}}{p}. \] (10)

Since \( w(\hat{n}) \leq v(n^*) = w(n_{\max}) \) and \( w(n) \) is increasing in \( n \), we shall have \( \hat{n} \leq \bar{n} \leq n_{\max} \) as shown in Figure 1.

4. A Dynamical Systems Model

The main objective of this paper is to examine whether these equilibria are stable (attracting) or unstable (repelling) and under what levels of controllable enforcement does the criminal activity tend to stabilize. We consider a certain type of criminal activity in an isolated neighborhood over a sequence of time periods. Potential criminals gather the information of actual risks and illegal monetary return from their own experience and the advice of peers. Decisions are then guided by a costs-benefits assessment. As rational decision makers, they only commit crimes when their predicted expected monetary return for crime exceeds their reservation wage, i.e., the “price” to enter the crime market.

Let \( n_t \) denote the number of crime incidents occurring during time period \( t \). These \( n_t \) number of crime incidents will experience an expected return of \( v(n_t) \) during this time period. With this realized value, criminals will make a prediction of the return, \( w_{t+1} \), expected for time
period \( t+1 \). Those criminals for whom this predicted expected return exceeds their reservation wage will enter the crime market in period \( t+1 \). Thus, the reservation wage of the marginal customer, \( w(n_{t+1}) \), will equal the predicted expected return \( w_{t+1} \). We suppose that criminals collectively predict the expected monetary return of crime, \( w_{t+1} \), using a convex combination of the previous estimate, \( w_t \), and the current return, \( v(n_t) \), via an exponential-smoothing forecast. Thus, using the fact that \( w(n_t) = w_t \) for all \( t \) gives the discrete dynamical system

\[
 w(n_{t+1}) = (1-u) w(n_t) + u v(n_t),
\]

with smoothing parameter \( u \). When \( 0 < u < 1 \), this prediction is known as adaptive expectations (cf. Carter and Maddock 1984), which is based upon a geometric weighting of the past observed expected returns \( v(n_0), v(n_1), ..., v(n_t) \). In the case of \( u = 1 \), potential criminals are assumed to have static expectations of the future based solely on current conditions. Thus, under static expectations the prediction, \( w(n_{t+1}) \), simply equals the current “price” \( v(n_t) \). The dynamics in this case lead to a so-called “cobweb” diagram as shown in Figure 2 (Samuelson, 1947).

Substituting Equations (3) and (4) into (7), we find an equivalent relation \( n_{t+1} = \Pi(n_t) \), where the operator \( \Pi(n) \) is given by

\[
 \Pi(n) = \ln \left[ (1-u) \exp(pn) + u c(m) \exp(-\alpha E/n-qn)/d(m) \right]/p \\
= \ln \left[ (1-u) w(n)/d(m) + u v(n)/d(m) \right]/p.
\]

We can view this dynamic process as a process whereby criminals seek an equilibrium number of crime incidents to balance their opportunity costs and the expected return from committing a crime. The fixed-point solutions to our dynamic system that solve \( n = \Pi(n) \) are the equilibrium points, \( \hat{n} \) and \( \tilde{n} \), that solve \( v(n) = w(n) \). The parameterization in \( \alpha \), \( \beta \) and \( E \) are useful when examining how the criminal behavior of the system depends on the arrest ability, crime type and the level of law enforcement, respectively.
5. Crime Equilibrium Stability

We now examine the asymptotic behavior of the dynamic system about the fixed-point equilibria. In particular, we examine issues of equilibrium stability and instability. A fixed-point, \( n^* \), is called *stable (unstable)* if the dynamical system converges to (diverges from) this point. A well-known (local) stability result says that the equilibrium \( n^* \) is stable (unstable) if \(|\Pi'(n^*)| < 1 \) \((|\Pi'(n^*)| > 1 \) (cf., e.g., Sandefur 1990; Devaney, 1986, Proposition 4). Therefore, we have the following results.

**Proposition 1.** The smaller crime equilibrium \( \tilde{n} = (\sqrt{k - \sqrt{k - 4\alpha\beta E}})/2\beta \) is unstable.

**Proof.** The first derivative of \( \Pi(n) \) is given by

\[
\Pi'(n) = \left[ (1-u) \frac{w(n)}{d(m)} + u \frac{v(n)}{d(m)} \right] \frac{1}{p} \left[ (1-u) \frac{w(n)}{d(m)} + u \frac{v(n)(\alpha E/n^2 - q)/d(m)}{p} \right].
\]

Using the equilibrium condition \( v(n) = w(n) = d(m) \exp(p\tilde{n}) \), we find

\[
\Pi' (\tilde{n}) = \exp(-p\tilde{n}) \left[ p(1-u) \exp(p\tilde{n}) + u \exp(p\tilde{n})(\alpha E/\tilde{n}^2 - q)/p \right]
= 1 + u(\alpha E/\tilde{n}^2 - q-p)/p
= 1 + u\beta \{ [\sqrt{4\alpha\beta E}/(\sqrt{k - \sqrt{k - 4\alpha\beta E}})]^2 - 1 \}/p > 1.
\]

The inequality follows from the fact that distinct equilibria exist if and only if \( k > 4\alpha\beta E \). However, the function \( \sqrt{k - \sqrt{k - 4\alpha\beta E}} \) (and hence the equilibrium) is decreasing in \( k \) with an upper bound over this range of \( \sqrt{4\alpha\beta E} \) achieved when \( k = 4\alpha\beta E \).

Proposition 1 says \( \tilde{n} \) is a repelling fixed-point; if the number of crime incidents is greater (less) than \( \tilde{n} \), the neighborhood attracts (detracts) crimes so that the crime level increases (decreases) away from \( \tilde{n} \). In this case the reservation wage curve \( w(n) \) is relatively flat, having smaller absolute slope than the expected return curve \( v(n) \) in a neighborhood of this equilibrium.
point, an equivalent statement of the stability condition (cf., e.g., Samuelson, 1947). Thus, in this neighborhood, \( w(n) \) is almost constant as assumed in Freeman et al. (1996).

Figure 3 illustrates this divergent “cobweb” diagram, showing that the crime diverges away from the smaller equilibrium. If the crime level is below \( \bar{n} \), it spirals monotonically down to zero. This is due to the fact that as the crime level decreases the amount of enforcement per crime increases, encouraging more criminals to leave. This phenomenon is called positive feedback (Kleiman, 1988, 1993), and will tend to collapse criminal activities in the neighborhood.

Since the smaller equilibrium \( \bar{n} \) is repelling, it will not be realized. We now turn attention to the larger crime equilibrium \( \bar{n} \). In contrast to the proof of Proposition 1, the larger equilibrium \( \bar{n} \) is increasing in \( k \). As a measure of the ratio between the proportionality constants for the value versus cost of crime, the parameter \( k \) is a measure of the “attractiveness” of crime. As this attractiveness parameter increases, so does the crime equilibrium \( \bar{n} \). The following proposition provides some information on the stability of the larger crime equilibrium \( \bar{n} \).

**Proposition 2.** When the smoothing parameter is sufficiently small, \( u < 2p/\beta \), then the larger crime equilibrium \( \bar{n} = (\sqrt{k} + \sqrt{k - 4\alpha E})/2\beta \) is stable. Otherwise, \( \bar{n} \) is stable under sufficient enforcement, \( E > E_b \), and unstable if \( E < E_b \), where \( E_b := k\gamma/\alpha(\beta + \gamma)^2 < E_c \) with \( \gamma := \beta - 2p/u \geq 0 \).

**Proof.** Following the proof of Proposition 1,

\[
\Pi'(\bar{n}) = \exp(-p\bar{n})(p(1-u) \exp(p\bar{n}) + u \exp(p\bar{n})(\alpha E/\bar{n}^2 - q))/p \\
= 1 + u(\alpha E/\bar{n}^2 - \beta)/p \\
= 1 + u(b^2 - \beta)/p,
\]

where
\[ b = 2\sqrt{\alpha E \beta / (k + \sqrt{k - 4\alpha \beta E})} < 2\sqrt{\alpha E \beta / \sqrt{k}} = (\sqrt{4\alpha \beta} / \sqrt{k})\sqrt{\beta} < \sqrt{\beta}. \]

Hence, the stability condition \(|\Pi'(\bar{n})| < 1\) holds if and only if \(\gamma = \beta - 2p/u < b^2 < \beta\). We immediately see that the latter stability inequality, \(b^2 < \beta\), is satisfied by the bound on \(b\), and the former inequality, \(b^2 > \gamma\), is trivially satisfied when \(\gamma < 0\), i.e., \(u < 2p/\beta\). Therefore, we now need only examine the condition \(b^2 > \gamma\) when \(\gamma \geq 0\). The condition \(b^2 > \gamma\) holds

\[ \iff 2\alpha \beta^2 E / (k - 2\alpha \beta E + \sqrt{k(k - 4\alpha \beta E)}) > \gamma \]

\[ \iff 2\alpha \beta (\beta + \gamma) E - k\gamma > \gamma \sqrt{k(k - 4\alpha \beta E)}. \]

Note that since the term on the right is strictly positive, if the term on the left is non-positive, i.e.,

\[ 2\alpha \beta (\beta + \gamma) E - k\gamma \leq 0, \]

then the condition fails and \(\bar{n}\) is unstable. Otherwise, if \(E > k\gamma/2\alpha \beta (\beta + \gamma)\), squaring both sides gives an equivalent necessary stability condition \(E > k\gamma/\alpha (\beta + \gamma)^2 = E_b\). Since \(\gamma = \beta - 2p/u < \beta\), this condition is also sufficient since

\[ k\gamma/\alpha (\beta + \gamma)^2 - k\gamma/2\alpha \beta (\beta + \gamma) = [k\gamma/\alpha (\beta + \gamma)][((\beta + \gamma)^{-1} - (2\beta)^{-1}) > 0. \]

The necessary and sufficient stability condition, \(E > E_b\), is not vacuous under the assumption of distinct equilibria, \(E < E_c\), since \(E_b = k\gamma/\alpha (\beta + \gamma)^2 < k\gamma/\alpha (4\beta \gamma) = k/4\alpha \beta = E_c\) when \(\gamma > 0\) and \(0 = E_b < E_c\) when \(\gamma = 0\). This completes the proof. \(\square\)

Proposition 2 concludes that \(u < 2p/\beta\) is a sufficient condition for the convergence of the system. When \(u \geq 2p/\beta\), then an amount of law enforcement exceeding \(E_b\) must be applied to ensure stability. As mentioned earlier, a crackdown enforcement exceeding \(E_c > E_b\) ensures a complete elimination of the crime. Clearly, a level of enforcement between \(E_b\) and \(E_c\) is possibly a more viable alternative. At this level, crime will not be eliminated, but at least contained.
Furthermore, the closer the enforcement can be increased toward $E_c$, the lower the crime equilibrium level $\hat{n}$.

It is interesting to note that $E_b$ is increasing with $u$ through the dependence on $\gamma$. This implies that, for larger values of the smoothing parameter $u$, more enforcement is necessary to maintain a stable level of crime. Put another way, stability under a static expectation implies stability under an adaptive expectation but not vice versa. Therefore, an unstable equilibrium under static expectations may be stable under adaptive expectations.

**Corollary 1.** If $p > q$, then $\hat{n}$ is a stable equilibrium.

**Proof.** If $p > q$, then $2p/\beta = 2p/(p + q) > 2p/(p + p) = 1$. But $0 < u \leq 1$, hence $u < 2p/\beta$ and the corollary follows by Proposition 2.

Recalling our previous discussions about parameters $p$ and $q$, $p$ reflects the level of criminals' opportunity costs and $q$ measures the level of victims' attention to the types of crimes. From Corollary 1, the criminal activities with high $p$ values and low $q$ values tend to stabilize easily. That is, criminal activities are more likely to be stable in situations that are in favor of criminals, i.e., criminals with higher opportunity costs or victims who are insensitive to crime.

**Corollary 2.** Neighborhoods that have more efficient arrest ability require lower levels of law enforcement to achieve stability of criminal activity.

**Proof:** The result is trivial due to Proposition 2 and the fact that $E_b$ decreases with $\alpha$.  

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Proposition 2 does not include the situation where the enforcement level is exactly $E_b$. In this case, the larger equilibrium $\tilde{n}_b$ is \( (\sqrt{k + \frac{1}{k} - \frac{4\beta\gamma k}{(\beta + \gamma)^2}}) / 2\beta \). The uncertainty of convergence about $\tilde{n}_b$ can be partially removed by making use of the so-called *Schwarzian derivative*

\[ \sigma := -2\Pi'''(\tilde{n}_b) - 3(\Pi''(\tilde{n}_b))^2. \]  

If $\sigma < 0$, then the equilibrium $\tilde{n}_b$ is stable; if $\sigma > 0$, then $\tilde{n}_b$ is unstable (cf. Sandefur 1990, Theorem 6).

The point $(\tilde{n}_b, E_b)$ is called a *bifurcation point*. In a bifurcation point, the equilibrium becomes unstable and the map often undergoes a *period doubling*. That is, the process initially cycles between two stable fixed points of $\Pi^2$. This process may continue as the enforcement drops further below $E_b$, developing oscillations between crime levels of period 4, 8, etc. The precise nature of a period-doubling bifurcation depends on the Schwarzian derivative as well as

\[ \eta := (\partial\Pi''/\partial E)(\tilde{n}_b) + 2(\partial\Pi'/\partial E)(\tilde{n}_b). \]

From bifurcation theory (cf., Lorenz 1989), $\eta \geq 0$ in our case since the equilibrium $\tilde{n}$ is stable for large enforcement levels $E$ and unstable for small $E$. This theory provides the following general proposition.

**Proposition 3.** Suppose $\eta \neq 0$ and $\sigma \neq 0$. If $\sigma < 0$, the system is stable for $E \geq E_b$ and becomes unstable for $E < E_b$ where a stable cycle of period doubling emerges. Otherwise, the system is stable for $E > E_b$ and diverges for $E \leq E_b$.  

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In the case of $\sigma < 0$, the bifurcation is called supercritical, i.e., the emerging cycles are stable, i.e., attracting. If the law enforcement applied in the neighborhood fails to meet the stability threshold $E_b$, then the number of crime incidents will oscillate from period to period between various crime levels. For $\sigma > 0$, the bifurcation is called subcritical, i.e., the emerging two-cycle is unstable. In this case, if the enforcement fails to exceed $E_b$, then the system is divergent and crime level diverges to the boundaries of an empty and full system with either 0 or $n_{max}$, respectively. These bifurcations are depicted in Figure 4.

Some studies (e.g., Gottfredson and Hirschi 1990; Wilson and Abrahamse 1992) suggest that, due to a lack of self-discipline, most criminals are thought to assess their costs-benefits based on immediate gains. That is, they have static expectations of the future based on solely on current conditions. The following corollary provides the information of criminal activity in this situation.

**Corollary 3.** Suppose criminals have static expectations of future crime value. If $p > q$, then the system is stable. Otherwise, the system is stable for sufficient enforcement $E \geq k(q-p)/4\alpha q^2$ and becomes unstable for $E < k(q-p)/4\alpha q^2$ where a stable cycle of period doubling emerges.

**Proof:** If $p > q$, then the system is stable by Corollary 1. Otherwise, for $u = 1$ our system is simplified as $\Pi(n) = (\sqrt{k} - \alpha E/n - qn)/p$. In this case, $E_b = k\gamma\alpha(\beta+\gamma)^2 = k(q-p)/4\alpha q^2$, and it is easy to check that $\sigma = -12\alpha E/p\tilde{n}_b^4 - 3(2\alpha E/\tilde{n}_b^3)^2/p < 0$. The result follows by Proposition 3. \qed
6. Summary and Conclusions

In this paper we formulated a mathematical model of rational criminal behavior criminal, modeling crime as an economic market with a supply of potential criminals who differ in their opportunity cost for committing crime. This heterogeneity among potential criminals generalizes the more frequent usage of a fixed reservation wage common to all criminals as assumed in Caulkins (1993) and Freeman et al. (1996). Put another way, the marginal “price” to enter the market, rather than being fixed, depends on the number of criminals; when the number of criminals is large the number of workers is low, raising the reservation wage for labor and hence the opportunity cost of crime. The market demand is influenced by the expected value for crime that, following the model of Freeman et al. (1996), is a product of the return for a successful crime, which depends on the level of crime, and the probability of success, i.e., not being arrested. The return function we propose includes several socio-economic variables including wealth, police enforcement and police arrest ability.

For this static market, there are either zero, one or two equilibrium points where the expected return for crime meets the reservation wage or opportunity cost of committing crime. The case of no such intersection corresponds to a market collapse with no crime. This case is achieved when the “ideal” expected return under an empty market with no police presence still is not sufficient to attract the most opportune potential criminal with the smallest assumed reservation wage. Clearly, this situation is not to be expected. The more realistic, but still highly idealized, instance under which the market collapses occurs when when a large police “crackdown” pressure is applied, where the enforcement exceeds a critical level $E_c$. The more realistic cases, with constrained enforcement resources $E \leq E_c$, involve the existence of two
equilibria, $\hat{n}$ and $\tilde{n}$, where $\hat{n} \leq \tilde{n}$ with equality and hence a single equilibrium in the case where enforcement equals $E_c$.

Finding these crime level equilibria suggested by this static model, we proposed a dynamical systems model in which criminals periodically respond to the crime level and its associated value bringing fluctuations in the level of crime. We assume that in one time period criminals experience an expected value for their criminal efforts that are broadcast to the population of potential criminals for consideration in following time periods. The criminals then collectively predict the value of crime for the next time period using an exponential smoothing forecast, which is simply a convex combination of the previous estimate and the current value. In economic theory this prediction corresponds to adaptive expectations (cf. Carter and Maddock 1984), which is based upon a geometric weighting of the history of observed crime returns.

Using this dynamical systems model, we studied the stability of this dynamical system. The larger equilibrium will be attracting provided the smoothing constant used by criminals is relatively small. Barring that, there exists an enforcement threshold, $E_b$. Exceeding this enforcement will push the crime to stabilize (converge) to this larger equilibrium. As mentioned above, additional enforcement beyond $E_c > E_b$ will collapse the crime market to zero. Failing to meet these critical levels with a relatively small but constant enforcement pressure may lead to a periodic fluctuation in the crime market over time, where the crime level either oscillates between several (period-doubled) values or diverges to the boundaries of an empty and full system with either 0 or $n_{max}$, respectively.

Such period-doubling behavior can lead to deterministic chaos, where stable periods of any size exist. Such phenomena has been explored in Rump and Stidham (1998) in the context of the design of a service facility. In fact, although we postulated explicit mathematical forms for
our crime market, the general unimodal nature of the expected crime value function $v(n)$ would suggest that the general qualitative findings here are robust (Chiarella, 1988).

7. Future Work

In building the crime value function in Section 2, we have followed the general model postulated by Freeman et al. (1996), in which the expected return from crime is a product of the return for a successful crime, which depends on the crime level, and the probability of success, i.e., not being arrested. If arrested, the criminal forfeits the return. A more general formulation would incorporate an additional monetary penalty (fine), say $f$, to be paid when arrested (cf., Becker, 1968; Sjoquist, 1973). Including this term will, in effect, transfer the fine to the opportunity cost, $w(n)$, to be “paid” in advance, adding an additional term $f \exp(-\alpha E/n)$ to “recoup” on a successful take $r(n)$. Like Freeman et al. (1996), we omitted this term to simplify the analysis. Certainly this omits certain crimes where such fines are imposed.

Leung (1995) has incorporated both monetary (fine) and temporal (jail term) penalty terms to create a dynamic programming model of deterrence that allows for individual criminal recidivism. Including these temporal penalties in our model and exploring the effects on market rather than individual forces would certainly be interesting future work.

It would be interesting also to examine the dynamic model presented here, now allowing for periodic changes in criminal behavior but also in the level of enforcement resources on part of the police. Stidham (1992) explored this idea in the context of the design of a queueing facility, where there the level of customer usage fluctuated about an equilibrium that depended on the service capacity resources.
The authors have used a version of the static model (under a fixed reservation wage) to study the allocation of a constrained set of enforcement resources across multiple neighborhood crime markets (Wang, 2000). There the problem boils down to a so-called knapsack problem (cf., e.g., Taha, 1975). This incorporates the notion of a spatial interaction and displacement of crime between neighborhoods (Rasmussen et al., 1993; Freeman et al., 1996). A similar analysis could be extended to the more general model here involving a heterogeneous population of potential criminals with different reservation wages. Using the insights in this paper, allocating at least an amount of neighborhood-specific enforcement $E_b$ can ensure the stability of criminal activity in that neighborhood. This provides us a baseline enforcement value for allocating enforcement in a neighborhood in order to control if not eliminate crime.

Acknowledgments

This research was supported by Grant No. 98-IJ-CX-K008, awarded by the National Institute of Justice, Office of Justice Programs, U.S. Department of Justice. Points of view in this document are those of the authors and do not necessarily represent the official position or policies of the U.S. Department of Justice.

We would like to thank the anonymous referees whose valuable suggestions drastically expanded the scope of this paper. We are also indebted to the editor for his encouragement to meet the challenge in expanding this paper’s scope.
References


Figure 1: Crime Equilibria

Monetary Return

Reservation Wage, \( w(n) \)

Expected Return, \( v(n) \)

Crime Level, \( n \)
Figure 2: A Cobweb Dynamic Model of Crime
Figure 3: The Unstable Equilibrium, $\tilde{n}$
Figure 4: Bifurcation Diagrams