Petroleum replenishment and routing problem with variable demands and time windows

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Abstract

In this paper we develop a methodological framework for designing the daily distribution and replenishment operations of petroleum products over a weekly planing horizon by taking into account the perspectives of both the transporter and its customers. The proposed approach considers the possibility of having late deliveries due to the variability of the customers' demands and expected time windows. We first develop an inventory model for the customers to identify the optimal order quantities and time windows. Then, we solve a sequence of mixed-integer optimization models for designing the distribution routes based on the order quantities and time windows selected by the inventory models. We designed the optimization models so that the late deliveries are balanced among the customers in order to mitigate the overall customer dissatisfaction. We test the proposed approach by solving a test bed of instances adapted from the literature. The empirical results show that the proposed approach can be used for designing the distribution plan for delivering petroleum products in conditions where the operational capabilities of the transporter are limited for generating optimal on-time plans.

Keywords: Petroleum delivery; Inventory routing; Routing and scheduling.

1 Introduction

The transportation industry plays a critical role in today's global economy fostering the operations of nearly all other industries around the world. Alone in the U.S., according to the U.S. Federal Highway Administration, about 20 billion tons of goods, worth more than $10 trillion, were moved across the country just in 2012 [47]. Transportation-related goods and services represented approximately 11% of the U.S. gross domestic product in 2000, only being surpassed by the housing, health-care, and food industries[20, 42].

Among the total goods transported in the U.S. in 2012, more than 1.8 billion tons corresponded to gasoline, diesel and other petroleum-based products, thus becoming the top sixth most transported commodities in the country [47]. Petroleum products are still one of the world's most traded commodities, as they continue being the main energy source for the transportation industry. According to the U.S. National Academy of Sciences [21], petroleum-based fuels represent about 98% of the energy sources used for mobilizing both people and freight in the U.S. In addition to their use as fuel for transportation, the need for petroleum products as lubricants, transmission and

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hydraulic fluids, and as raw materials for many production processes, has made of them ubiquitous and vital for the daily operations of every country.

In order to provide a steady supply of petroleum products that satisfies the growing demand, many suppliers face the enormously complex process of planning the routing strategies for distributing these products to their customers, which include gasoline retailers (gas stations), production companies, agricultural companies, and marine centers, among others. This process generally requires the allocation of multiple competing resources while simultaneously satisfying many operative restrictions and regulatory policies (the transportation of gasoline and other petroleum-based products is highly regulated by the Federal Motor Carrier Safety Administration (FMCSA) in the U.S. [31]). As a result, since publication of the first paper on gasoline distribution [27], there has been a continuous effort to develop quantitative models that support the decision making of transportation companies.

Over the last couple of decades numerous solution approaches have been developed to tackle vehicle routing problems that incorporate the distribution requirements of many industries (e.g., [3, 6, 14, 23, 49, 50]). However, despite the large number of such approaches, there are still several elements of the distribution logistics of some products that require further analysis. One fundamental assumption that is often considered by many of these technologies is that there always exist distribution plans that satisfy all the customers' requirements (e.g., order quantities and time windows) under the transporter operational restrictions.

In a real-life scenario, the decisions regarding the product orders and the distribution logistics are often made independently and sequentially by the customers and the transporter without any interaction during the distribution planning process. In other words, the order quantities and expected delivery time windows—which are selected by each customer solely based on the customer's own interests—are given to the transporter in the form of hard constraints. Then, after collecting all the orders, the transporter makes a routing plan aiming to fulfill all the requirements while minimizing its own operational costs [2, 49]. Many mathematical models assume that there are always feasible routes that satisfy such demands and time requirements of all the customers. In contrast, in many competitive markets like the one of products distribution—in which the transportation decisions must also meet strict regulatory policies—finding feasible solutions that cope with all customer and governmental requisites is often impossible. In other words, due to capacity limitations and further operational requirements, the actual delivery time may deviate from the desired time window for many customers.

For the specific case of petroleum products, there are several reasons why optimally planning the distribution logistics is a complex challenge. In addition to the variable nature of the demands, the limited number of trucks and drivers, the regulatory policies, and the difficulties posed by the inherent characteristics of the products (i.e., mostly flammable liquids that must be transported in specialized multi-compartmented trucks and trailers), the heterogeneity of the customers and the strict time requirements make the problem of identifying optimal delivery routes especially difficult. Moreover, because of the replenishment logistics, most customers request very specific delivery time windows that often overlap among them (e.g., retailers often prefer having replenishments late at night when traffic is low, whereas other customers prefer early morning deliveries before any operation begins). These latter requirements dramatically impact the complexity of the distribution planning, up to the point that even finding feasible delivery schedules is simply impossible.

1.1 Relevant literature

The *vehicle routing problem* (VRP) is at the cornerstone of most distribution planning processes. Since the publication of the first paper in the subject back in the late 50's [27] (a paper about gasoline distribution), a staggering number of studies have been developed to tackle many variants of this problem (for further references see the following comprehensive surveys [22, 29, 49]). This problem and its variations have continuously raised the interest of the academic community because of their practical relevance and inherent difficulty. In fact, several technological advancements in the field of Operations Research have been discovered by studying these particular problems [22].

Among all the variations that can be found in the literature, the ones that are relevant for this paper are the VRP with time windows (VRPTW), where each customer must be visited during a specific time frame [4, 9, 10, 33, 36]; the VRP with multiple compartments (MCVRP), where the vehicles have different capacities and are equipped with multiple compartments that can carry more than one type of product [1, 15, 16, 18, 28, 30, 34, 35, 37, 40, 44]; the VRP with stochastic demands (SVRP), where the demands are given by a probability distribution [7, 8, 17, 48, 38]; the dynamic VRP (DVRP), where the information of some customers becomes available during operation [43]; the time window assignment VRP (TWAVRP), which is a variation where the time windows have to be assigned by the transporter to the customers before the demand of product given by them is known [45, 46]).

In addition to the aforementioned problems regarding vehicle routing, an alternative type of approach that has been consistently utilized to plan the distribution logistics of several products is that of synchronizing the inventory management with the design of the distribution logistics of the commodities [2, 13, 19, 20, 32, 39]. The resulting *inventory routing problems* (IRPs) can produce several advantageous strategies if the customers are willing to delegate the inventory management.

In general, the IRP is a product distribution problem in which one actor—the *manager*—is responsible for both transportation and inventory planning [19]. In practice, the manager can either be the producer, the consumer, or the transportation company depending on the type of business. When the managers are the producers or the transportation companies—which is often the case—these integrated policies allow them to select the timing and sizes of the deliveries, achieving a better utilization of their vehicle fleet and offering a better service quality to their customers. For these models to be applicable though, the customers must render complete knowledge of their operational needs and full control over their inventory levels to the manager. In turn, the manager must ensure that the customers will never out of stock. Nevertheless, in the context of petroleum products, the operational decisions of many customers require them to maintain full control of the inventory levels, which complicates the use of these latter models. Thus, the main difference between the problem studied in this paper and the IRP is that in the latter the suppliers control the inventory management of the customers, whereas in the former, the inventory problem is solved by each customer and the results are given as hard constraints to the transporter.

For the specific context of delivering petroleum products, the first publications that provide specific applications of this kind date back to the 50's [27], 80's [11, 12] and 90's [5, 50]. Most of the solution approaches proposed in these papers range between heuristic and exact approaches. For instance, [3] formulated a fuel delivery problem as a set partitioning model and proposed a branch-and-price algorithm to solve the resulting problem—a technique widely used for solving VRPs. In addition to that, [41] presented a case study on the delivery networks in Hong Kong, which contains tanker assignment and a routing problem with a heterogeneous fleet of compartmented trucks. A decision support system (DSS) approach was developed to solve the vendor managed inventory (VMI) problem. In [23], an exact algorithm was proposed to tackle the single period and single depot case using an unlimited heterogeneous fleet of compartmented tank trucks petroleum replenishment problem. A heuristic for the multi-period and single depot with limited number of trucks was proposed in [24]. The same problem with time window constraints was tackled further in [26]. More recently, [25] proposed heuristics for the multi-depot station replenishment problem considering time windows, in which the concept of a trip—defined by both a route and the truck used to make deliveries in this route—was firstly introduced to address this problem. Instead of generating possible routes, they introduced a method to generate potential feasible trips. Similarly, as for other VRP variants, the time windows given by the customers are assumed to be fixed and no further considerations are proposed for the cases where the given time windows render the problem infeasible.

1.2 Contributions

This paper aims to develop a methodological framework for designing the daily distribution operation for delivering petroleum products that considers the possibility of having late deliveries due to the variability of the customers demands and expected time windows. In addition to maximizing the distribution profits, the proposed framework attempts to minimize the dissatisfaction of the customers due to late deliveries by balancing the late deliveries among the customers over the planning horizon. The contributions of this paper can be summarized as follows:

- We propose a methodological framework to solve the daily petroleum distribution problem considering both transporter's and customers' perspectives.
- We develop an inventory problem that models all the scenarios regarding the delivery times for the product orders. This model is used to determine the order quantities and time windows for each customer.
- We propose a sequence of mixed-integer optimization models for designing the distribution routes based on the order quantities and time windows selected by the inventory models.
- We tailor the optimization models so that the late deliveries are balanced among the customers in order to mitigate the overall customers' dissatisfaction.
- We test the proposed approach by solving a test bed of instances adapted from the literature. The empirical results show that the proposed approach can be used for designing the distribution plan for delivering petroleum products in conditions where the operational capabilities of the transporter are limited for generating optimal on-time plans.

The remainder of this paper is organized as follows. Section 2 presents a detailed description of the problem at hand; Section 3 summarizes of the proposed sequential solution approach; Section 4 introduces the inventory model that is used to model the gas station decisions; Section 5 present the proposed mathematical formulations to generate the distribution plan; Section 6 analyzes the results of the empirical study; and finally, Section 7 provides the final conclusions and further research directions.

2 Problem description

The petroleum replenishment problem deals with the logistics of delivering petroleum products to a set of customers—in this case gas stations—in such a way that the requirements of such customers are fulfilled, under the operational capabilities of the transporter, while maximizing the total distribution profits yielded by the distribution operation. From the perspective of the customers, during their daily operations, gas stations periodically review their underground tanks and place order requests to the suppliers when the stock levels of their products fall below predefined thresholds. The orders typically consist of a list of product type requests that include the desired quantities and delivery time windows—as shown in Figure 1. The supplier collects this information from all of its customers to generate: (1) the truck loading, (2) the delivery routes, and (3) truck distribution schedule, in order to fulfill these orders. From the perspective of the supplier, the distribution network is defined as follows. Let $G = (N, A)$ be a directed graph where $N = \{0, 1, 2, .., n\}$ is a set of nodes representing the distribution terminal (node 0) and the gas stations (nodes $1, .., n$), and $A = \{(i, j) : i \neq j \text{ and } i, j \in N\}$ is the set arcs that represent the road segments connecting the nodes in N. We denote $c_{ij}(c_{ij} = c_{ji}$ and $c_{ij} > 0)$ and t_{ij} as the travel costs and travel times associated with the arc (i, j), and s_i as the service time of gas station i. The time window $[a_i, b_i]$ specifies the earliest and latest time limits for performing the petroleum replenishment, i.e., the delivery must occur within the given time window $[a_i, b_i]$.

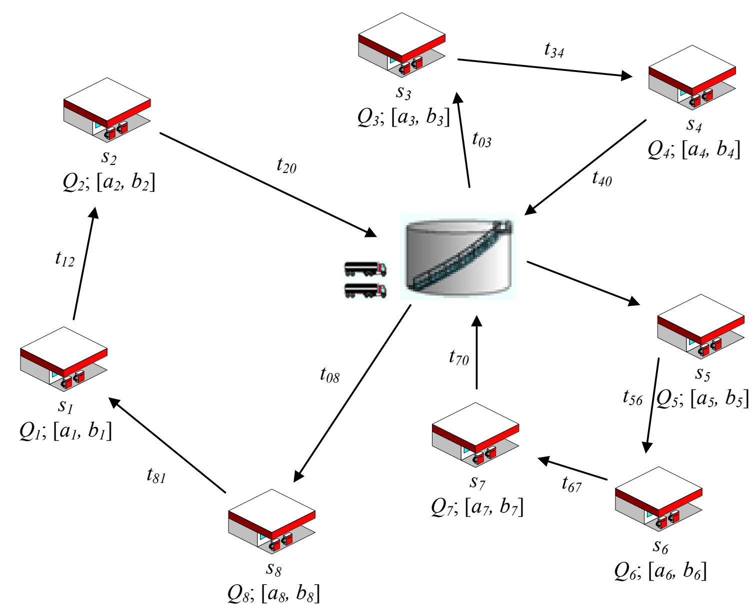


Figure 1: Distribution network.

Each truck is divided into multiple compartments with known capacities, which are used to upload different type of products. In other words, two distinct grades of petroleum must be placed into two separate compartments to avoid internal contamination. Furthermore, the petroleum stored in each compartment must be fully pumped out when fulfilling the distribution service, as the quality of any remaining petroleum will deteriorate once it has contact with air, due to oxidation. Therefore, if the underground tanks of a gas station fail to accommodate a full-compartmented load of petroleum, the remainder must be sent back to the terminals, resulting in a send-back cost—which is generally high in comparison to other costs. All trucks should begin and end at the terminal and the travel speed of those are considered to be the same. Thus, the petroleum replenishment problem consists of determining:

(1) the quantity and time windows of delivery for each gas station;

(2) the loading of the various petroleum into the truck compartments;

(3) the delivery routes to the gas stations;

(4) the departure time of each truck from the terminal and the arrival time at each of its assigned customers.

In addition, the objective of the distribution problem is twofold: to minimize the expected total costs for gas stations, and to maximize the overall distribution profits for the transporters. In this paper we decouple the petroleum delivery problem into two parts: the gas station inventory problem (i.e., part (1) of the above list) and the transporter distribution problem (i.e., parts (2)-(4)). First, for the gas station inventory problem, we use a model to determine the order quantities and delivery time window for each gas station and second, for the transporter's distribution problem we propose a sequence of mixed-integer formulations to determine the distribution plan. The description of the proposed framework is given in the following section.

3 Solution framework

The petroleum replenishment problem deals with both the inventory problem of the gas stations and the distribution problem of the transporter on a daily basis. For each day of the planning horizon, the order quantities and requested time windows for each gas station will be identified by an inventory model that is aimed to minimize the expected total costs perceived by the gas stations. Such costs include ordering cost, holding cost, shortage cost, and send-back cost (see Section 4). The proposed model takes into consideration that the exact delivery times are not known a priory by the gas stations. Therefore, the order quantities and desired time windows are decided based on an estimation of the delivery times.

Once the order quantities and desired delivery time windows are selected by each gas station using the proposed inventory model, we then solve a series of interrelated mixed-integer linear programs that will determine how to load these demands into the truck compartments, how to schedule the truck departure and returning times, and how to deliver these demands (see Section 5). By solving these models, the optimal petroleum distribution plan for each day can be identified.

In the proposed approach, we solve the replenishment problem sequentially day by day. Thus, the resulting distribution plan obtained for each day of the planing horizon is then used to calculate the inputs of the inventory models of the subsequent days. Notice that, after solving the distribution problem of a given day, the actual delivery times for each station can be used to compute the initial stock levels for each station for the day after and thus, the order decision can be made accordingly. The solution process continues until the considered time horizon is reached. In this section, the steps of the proposed approach, which are presented in Figure 2 are summarized.

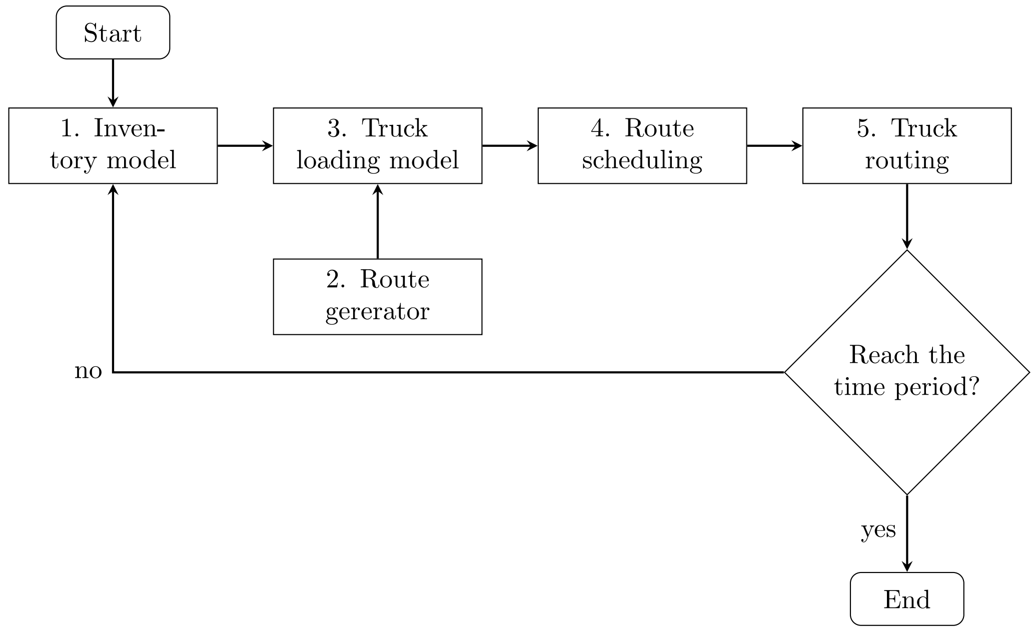


Figure 2: Solution procedure.

- *Step 1: Inventory model*

  The first step of the solution framework is to generate the product orders of each gas station. The information required to generate the orders in this step comprises the initial stock levels, the demand rates, and the tank capacities. In addition to the gas station information, the inventory related costs are also required to calculate the expected total costs. These costs include the unit order cost, fixed order cost, holding cost, shortage cost, and send-back cost. The inventory model is used to determine the order quantities and desired delivery time windows that minimizes the gas stations expected total costs based on their stock levels and petroleum consumption rates. A full description of this model is given in Section 4.

- *Step 2: Route generator*

  The delivery routes are generated for the gas stations that placed orders during the given day according to the inventory model. Typically, a truck contains four to six compartments with different capacities and each gas station requires one or two compartments to satisfy the petroleum requirement. Therefore, having routes serving between one and three stations is common in practice. However, we also consider the situation where the petroleum distribution serves gas stations with lower demand rates. For this case, trucks can visit four to five stations within a route. We describe the process used to generate the candidate routes in Section 5.1.

- *Step 3: Truck loading capacity check*

  The total delivery quantities of a route cannot exceed the capacities of the compartments in the truck. The truck loading model is used to determine the assignment of the different petroleum products to each of the truck compartments. The route will be eliminated if the demands of the gas stations of the given route cannot be loaded into the truck. In addition, this model computes the profit of the route, which is calculated by the revenue received for delivering the petroleum minus the travel costs of the route.

- *Step 4: Route scheduling*

The truck schedule of each feasible route generated is then determined in this step. The objectives are to find the truck departure and returning times, the delivery times for each gas station in the route, and a set of penalties for those routes failing to satisfy the time window constraints. A route that fails to satisfy the time window constraints is not eliminated in this step. Instead, we add a penalty that accounts for the total time the truck following the given route arrives before or after the gas station time windows. Consequently, the generated routes are divided into two sets: feasible (on time) routes and infeasible (late) routes and their costs are updated with the corresponding penalties.

- *Step 5: Truck routing*

  In the truck routing problem, the truck assignment will be decided by maximizing the total profits of delivering the petroleum to all gas stations. The objective calculation includes the route profit found in Step 3, as well as the penalties for late deliveries obtained in Step 4.

- *Step 6: Termination condition*

  As mentioned before, the petroleum distribution problem is solved sequentially on a daily basis. The results of route scheduling and truck routing for day t are inputs for day $t+1$. If the considered time horizon is not reached, then return to Step 1 to resolve the problem for the subsequent day. If the time horizon is reached, we terminate the process.

4 The inventory model for the gas stations

The inventory model seeks to determine the desired delivery time windows $[a_i, b_i]$ over the time horizon T and the order quantities q_i for each customer $i \in N$ so that the expected customer's total cost is minimized. Let l_i be the actual delivery time for customer $i \in N$. The total cost of each customer given delivery time l_i is named $C_i(l_i)$ and comprises: (1) the ordering cost $P_i(q_i)$, which represents the cost of ordering product to the supplier; (2) the holding cost $H_i(l_i, a_i, b_i, q_i)$, which is the opportunity cost of having inventory; (3) the shortage cost $S_i(l_i, a_i, b_i, q_i)$, which is the cost the customer incurs if at some point it runs out of inventory (e.g., for the case of a retailer, the equity cost associated with losing potential sales); and (4) the send-back cost $B_i(l_i, a_i, b_i, q_i)$, which is a monetary penalty payed to the transporter for ordering product in excess (i.e., more than what the customer can accommodate at delivery time). Notice that the order quantities and time windows are selected by each customer before the transporter decides upon the distribution planing. Hence, from the perspective of the customer, the exact delivery time l_i is not known when solving its inventory problem. Thus, the total cost for customer i given as a function of the delivery time l_i is:

$$C_i(l_i) = \min_{[a_i,b_i]\in T, q_i\geq 0} P_i(q_i) + H_i(l_i, a_i, b_i, q_i) + S_i(l_i, a_i, b_i, q_i) + B_i(l_i, a_i, b_i, q_i). \tag{1}$$

The ordering cost P_i is often given by a function including a fixed ordering cost k plus the product between the order quantity q_i and a unitary price per gallon F. We will assume that the order quantity is made so that upon arrival, the transporter delivers enough gasoline to fill the underground tanks of the gas station. Therefore, such an order quantity depends on the initial stock r_i, the demand rate d_i, and the expected delivery time. Ideally, the gas station would expect the product to be delivered at the midpoint $(a_i + b_i)/2$ of the time window, as this would minimize possible shortages or send-backs. Nevertheless, since the gas station does not know in advance the actual delivery time l_i—as this is decided by the transporter after all the orders are collected—the gas station must then base its decisions on the idea that the delivery truck will arrive at any time within the time window $[a_i, b_i]$. For this reason, from the perspective of the gas stations, the delivery time $l_i \sim U[a_i, b_i]$ is a random variable uniformly distributed over the interval given by a_i and b_i. Furthermore, since the expected delivery time is the midpoint $(a_i+b_i)/2$, the order quantity is therefore equal to the capacity of the underground tank minus the amount of left at $(a_i + b_i)/2$.

In general, shortage costs are generally more expensive than the holding costs. Thus, the customers aim to select time windows so that the time to empty T_i^e, also referred to as the stock out time (i.e., the expected time in which the customer runs out of inventory), occurs after the window upper bound b_i.

To properly model the above inventory problem, there are several cases that must be considered depending on the values of a_i, b_i, q_i, l_i, and T_i^e. Particularly, if for the given values of $[a_i, b_i]$, the time to empty T_i^e falls either: (1) before a_i, (2) between a_i and b_i or, (3) after b_i.

The following assumptions are made for the gas station inventory model:

- for each demanded product, the gas station solves an inventory model to determine $[a_i, b_i]$ and q_i;
- the delivery time l_i is assumed uniformly distributed over the interval $[a_i, b_i]$;
- the time horizon is discretized hourly;
- 12 working hours are considered for one working day;
- no backlogging of demand is allowed (i.e., sales not met due to shortages are lost);
- the demand rate of each gas station is assumed to be constant and known;
- inventory is continuously reviewed and all replenishment decisions are made at the beginning of each time period;
- each gas station has a known starting level r_i and that information is not disclosed to the supplier;
- once the orders are placed, those cannot be changed;
- the order quantity q_i is equal to the capacity of the underground tank minus the amount left at the expected arrival time;
- the stock out time of each station T_i^e is known (since the demand rate and the initial stock are known, the time to empty can be calculated in advance);
- the possible time window choices are discrete.

All the mathematical notation for the inventory model are given in Table 1.

Table 1: Mathematical notation for the inventory model.

Notation	Definition
Q_i	The underground tank capacity of station i
T_i^e	Stock out time of station i
d_i	Demand rate of station i
l_i	Delivery lead time which is unknown by the customers as the routing schedule is generated after all the orders are known and it can be anywhere between a_i and b_i. It is assumed to follow a uniform distribution with limits a_i and b_i
r_i	Initial stock level of underground tank of station i
F	The order cost per gallon
k	Fixed cost per delivery
h	Holding cost (per unit per unit time)
s	Shortage cost (per unit per unit time)
p	Send-back cost (per unit per unit time)
a_i	Earliest time window of station i
b_i	Latest time window of station i
q_i	Order quantity of gas station i

When gas stations receive the orders, three scenarios may occur:

- Scenario 1: the stock out time of a station T_i^e occurs prior to a_i.
- Scenario 2: the stock out time of a station T_i^e occurs within time window $[a_i, b_i]$. This scenario comprises two cases: (1) both the lead time l_i and stock out time T_i^e occur before the midpoint $(a_i + b_i)/2$, (2) both the lead time l_i and stock out time T_i^e occur after the midpoint $(a_i + b_i)/2$. In other words, we have the possibilities $a_i \leq l_i \leq T_i^e \leq (a_i + b_i)/2 \leq b_i$ and $a_i \leq T_i^e \leq l_i \leq (a_i + b_i)/2 \leq b_i$ in case 1; and $a_i \leq (a_i + b_i)/2 \leq l_i \leq T_i^e \leq b_i$ and $a_i \leq (a_i + b_i)/2 \leq l_i \leq T_i^e \leq b_i$ in case 2.
- Scenario 3: the stock out time of the station T_i^e occurs after b_i, which comprises two cases as well: (1) the petroleum is received before the midpoint of the time window ($l_i \leq (a_i + b_i)/2$) or, (2) the petroluem is received after the midpoint of the time window ($l_i \geq (a_i + b_i)/2$).

These three scenarios are summarized in Table 2 and the corresponding expected total cost calculations are presented afterwards.

Table 2: Summary of scenarios.

	Stock out time	Case in scenario
Scenario 1	$T_i^e < a_i$	
Scenario 2	$a_i \leq T_i^e \leq b_i$	Case 1a: $a_i \leq l_i \leq T_i^e \leq (a_i + b_i)/2 \leq b_i$
		Case 1b: $a_i \leq T_i^e \leq l_i \leq (a_i + b_i)/2 \leq b_i$
		Case 2a: $a_i \leq (a_i + b_i)/2 \leq l_i \leq T_i^e \leq b_i$
		Case 2b: $a_i \leq (a_i + b_i)/2 \leq T_i^e \leq l_i \leq b_i$
Scenario 3	$b_i < T_i^e$	Case 1: $l_i \leq (a_i + b_i)/2$
		Case 2: $l_i \geq (a_i + b_i)/2$

- Scenario 1: $T_i^e \leq a_i$

  In scenario 1, the stock out time of a station T_i^e occurs prior to a_i. This implies that there will be a cost associated with a shortage of petroleum. Consequently, the send-back cost is 0 and the order quantity q_i is equal to the underground tank capacity Q_i. Figure 3 provides a graphical representation of the inventory of scenario 1.

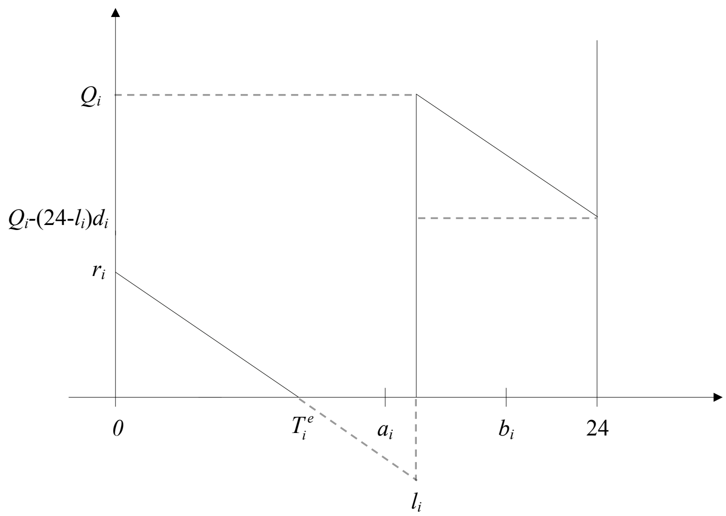


Figure 3: The inventory model of scenario 1.

The ordering cost is given by fixed cost plus variable cost, then

$$k + FQ_i,$$

where k is the fixed cost and F is the unit order cost per gallon. The expected daily holding volume can be calculated from the inventory function depicted in Figure 3. Thus, the holding volume before the underground tank becomes empty is $(r_i T_i^e)/2$ and the expected daily holding volume after order arrival is $Q_i(24 - l_i) - [(24 - l_i)^2 d_i]/2$.

Therefore, we obtain the expected daily holding cost

$$\left[\frac{r_i T_i^e}{2} + Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right] h$$

The total shortage cost is given by

$$\left[\frac{(l_i - T_i^e)^2 d_i}{2}\right] s$$

which is average shortage volume times the shortage cost.

The send-back cost is 0 because the underground tank will become empty prior to receiving the ordered petroleum and thus it can accommodate the full order. Hence, the total cost for the scenario 1 denoted by $C_i^{S1}(l_i)$ is then:

$C_i^{S1}(l_i) =$ ordering cost + holding cost + shortage cost + send-back cost

$$= (k + FQ_i) + \left[\frac{r_i T_i^e}{2} + Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right] h + \left[\frac{(l_i - T_i^e)^2 d_i}{2}\right] s \tag{2}$$

To calculate the expected cost, we take the integral over a_i and b_i for l_i. Thus, the expected total cost of scenario 1, $E[C_i^{S1}]$, is given by:

$$\begin{aligned}
E[C_i^{S1}] &= \int_{a_i}^{b_i} C_i^{S1} \frac{1}{b_i - a_i} dl_i \\
&= \frac{1}{6}[-3(-48 + a_i + b_i)hQ_i + 6(k + FQ_i + 3hr_i T_i^e + d_i[-(1728 + a_i^2 \\
&\quad + a_i(-72 + b_i) + (-72 + b_i)b_i)h + s(a_i^2 + a_i b_i + b_i^2 - 3(a_i + b_i)T_i^e + 3(T_i^e)^2))]
\end{aligned} \tag{3}$$

- Scenario 2: $a_i \leq T_i^e \leq b_i$

  In scenario 2, the stock out time of a station occurs within the time window $[a_i, b_i]$. It should be noted that the time differences among l_i, T_i^e, and $(a_i + b_i)/2$ within $[a_i, b_i]$ play a significant role in evaluating the expected total cost. There will be a send-back cost and no shortage cost if a station receives the order prior to running out of petroleum, which is the situation given by $l_i < T_i^e$. Conversely, there will be a shortage cost and no send-back cost if a station runs out of product before receiving petroleum ($l_i \geq T_i^e$). Thus, we consider these two cases in scenario 2:

  - Case 1: the petroleum is received before the midpoint of the time window ($l_i \leq (a_i + b_i)/2$)

    a. $a_i \leq l_i \leq T_i^e \leq (a_i + b_i)/2 \leq b_i$

    As depicted in Figure 4(a), gas station i will not run out of petroleum, but it will incur in a send-back cost. The expected daily holding volume before receiving petroleum is $r_i l_i - (l_i^2 d_i)/2$ and the expected daily holding volume after the order arrival is $Q_i(24 - l_i) - [(24 - l_i)^2 d_i]/2$. Hence, the daily holding cost is:

    $$\left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + \left(Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right)\right] h,$$

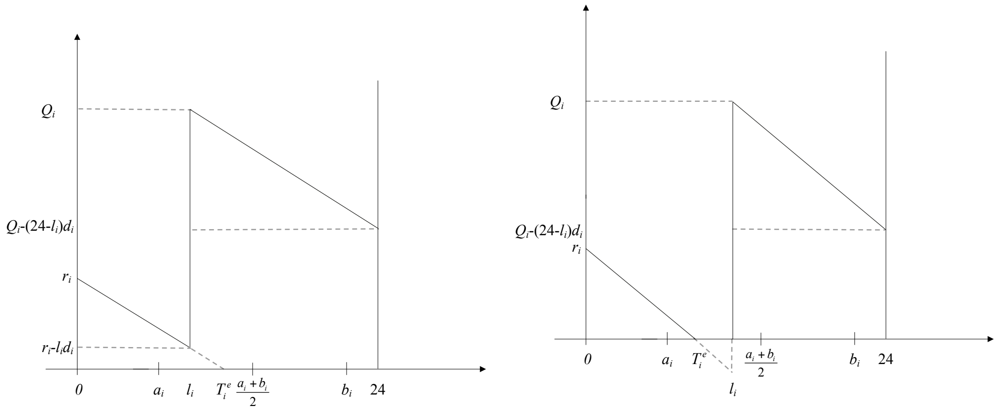
    Since the time to empty occurs before the midpoint of the time window, the order quantity q_i is equal to the underground tank capacity Q_i. Also, the quantity that exceeds the tank capacity when receiving petroleum is $r_i - l_i d_i$, so the send-back cost is $(r_i - l_i d_i)p$. Therefore, the total cost for this case denoted by $C_i^{S2C1a}(l_i)$ is:

    $$\begin{aligned}
    C_i^{S2C1a}(l_i) = (k + FQ_i) + &\left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + \left(Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right)\right] h+ \\
    &+ (r_i - l_i d_i)p
    \end{aligned} \tag{4}$$

    b. $a_i \leq T_i^e \leq l_i \leq (a_i + b_i)/2 \leq b_i$

    In contrast, a shortage cost $[(l_i - T_i^e)^2 d_i]/2$ and send-back cost 0 will occur in scenario 2 case 1b, as shown in Figure 4(b). For this case, the expected daily holding cost is

    $$\left[\frac{r_i T_i^e}{2} + (24 - l_i)Q_i - \frac{(24 - l_i)^2 d_i}{2}\right] h,$$



(a) Case 1a (b) Case 1b

Figure 4: The inventory model of scenario 2 case 1.

where $(r_i T_i^e)/2$ represents the expected daily holding volume before running out of petroleum and $Q_i(24 - l_i) - [(24 - l_i)^2 d_i]/2$ is the volume after receiving ordered petroleum. The average shortage volume is $[(l_i - T_i^e)^2 d_i]/2$ and therefore the total shortage cost is

$$\frac{(l_i - T_i^e)^2 d_i}{2} s.$$

Let $C_i^{S2C1b}(l_i)$ be the total cost for scenario 2 case 1b. That is,

$$C_i^{S2C1b}(l_i) = (k + FQ_i) + \left[\frac{r_i T_i^e}{2} + (24 - l_i)Q_i - \frac{(24 - l_i)^2 d_i}{2} \right] h + \frac{(l_i - T_i^e)^2 d_i}{2} s \tag{5}$$

Similarly as for case 1, to calculate the expected cost, we take the integral over the interval from a_i to T_i^e for (4) and from T_i^e to b_i for (5). The expected total cost of scenario 2 case 1, $E[TC_i^{S2C1}]$, is given by:

$$\begin{aligned} E[C_i^{S2C1}] &= \int_{a_i}^{T_i^e} C_i^{S2C1a}(l_i) \frac{1}{b_i - a_i} dl_i + \int_{T_i^e}^{b_i} C_i^{S2C1b}(l_i) \frac{1}{b_i - a_i} dl_i \\ &= \frac{1}{6(a_i - b_i)} [-2a_i^3 d_i h + 3a_i^2 (24 d_i h - d_i p - hQ_i + hr_i) + 6a_i(k + FQ_i \\ &\quad + 24h(-12d_i + Q_i) + pr_i] + b_i[(1728 + (-72 + b_i)b_i)d_i h + 3(-48 + b_i)hQ_i \\ &\quad - 6(k + FQ_i) - b_i^2 d_i s) + 3(-b_i h r_i - 2pr_i + b_i^2 d_i s)T_i^e + 3d_i(p - b_i s)(T_i^e)^2 \\ &\quad + d_i(h + s)(T_i^e)^3] \end{aligned} \tag{6}$$

– Case 2: the petroleum is received after the midpoint of the time window ($l_i \geq (a_i + b_i)/2$)

a. $a_i \leq (a_i + b_i)/2 \leq l_i \leq T_i^e \leq b_i$

As mentioned before, the order quantity of station i is set to the amount required to fill the underground tank at the time $(a_i + b_i)/2$. In case 2a, since the underground tank will not become empty before the order expected arrival time $((a_i+b_i)/2 < T_i^e)$, the order quantity will not be equal to the full capacity of the tank, as depicted in Figure 5(a). For this case, the order quantity is given by:

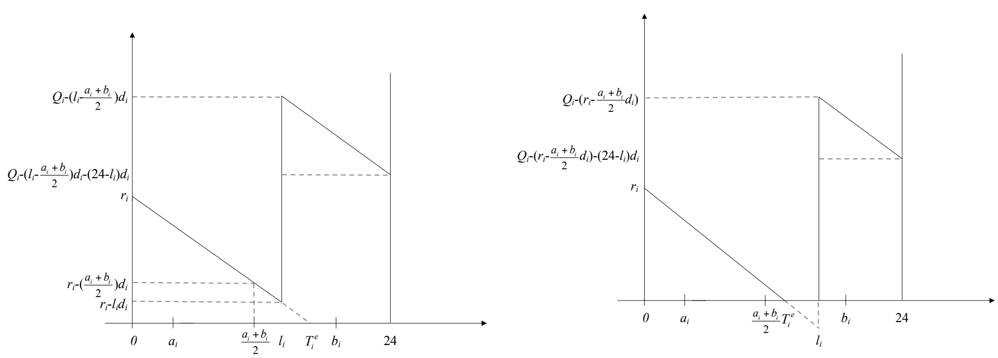
$$Q_i - \left[r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right], \tag{7}$$

where $r_i - [(a_i + b_i)/2]d_i$ represents the quantity left in the tank at the expected arrival time. The expected daily holding volumes before and after the order arrival are $r_i l_i - (l_i^2 d_i)/2$ and $(24 - l_i)[Q_i - [l_i - (a_i + b_i)/2]d_i - [(24 - l_i)^2 d_i]/2]$, respectively. The expected daily holding costs are:

$$\left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + (24 - l_i)\left(Q_i - \left(l_i - \frac{a_i + b_i}{2}\right) d_i\right) - \frac{(24 - l_i)^2 d_i}{2}\right] h.$$

Both the shortage cost and send-back cost are 0 since the stations will receive the petroleum before they run out and the ordered quantity will not exceed the tank capacity because the actual order arrival time occurs after the expected arrival time $(a_i + b_i)/2$. Let $C_i^{S2C2a}(l_i)$ denote the total cost for scenario 2 case 2a. That is,

$$\begin{aligned} C_i^{S2C2a}(l_i) = & \left[k + F\left[Q_i - \left[r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right]\right]\right] \\ & + \left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + (24 - l_i)\left(Q_i - \left(l_i - \frac{a_i + b_i}{2}\right) d_i\right) - \frac{(24 - l_i)^2 d_i}{2}\right] h \end{aligned} \tag{8}$$



(a) Case 2a

(b) Case 2b

Figure 5: The inventory model of scenario 2 case 2.

b. $a_i \le (a_i + b_i)/2 \le T_i^e \le l_i \le b_i$

The ordering costs in case 2b are also given by expression (7), as T_i^e occurs after the midpoint $(a_i + b_i)/2$. Additionally, the holding cost is given by:

$$\left[\frac{r_i T_i^e}{2} + \left[Q_i - \left(r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right)\right](24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right] h,$$

where $(r_i T_i^e)/2$ represents the expected daily holding volume before running out of petroleum and

$$\left[Q_i - \left(r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right)\right](24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}$$

is the volume after receiving the petroleum.
The average shortage volume is $[(l_i - T_i^e)^2 d_i]/2$ and the total shortage cost is then given by

$$\frac{(l_i - T_i^e)^2 d_i}{2} s$$

Also, the send-back cost is 0 since the tanks are always able to accept the full order in this case. Thus, the total cost for scenario 2 case 2b is then:

$$\begin{aligned} C_i^{S2C2b}(l_i) &= \left[k + F\left[Q_i - \left[r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right]\right]\right] \\ &+ \left[\frac{r_i T_i^e}{2} + \left[Q_i - \left(r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right)\right](24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right] h \\ &+ \frac{(l_i - T_i^e)^2 d_i}{2} s \end{aligned} \tag{9}$$

Furthermore, taking integral over the range a_i to T_i^e for case 2a and over T_i^e to b_i for case 2b after combining (8) and (9), the expected total cost of scenario 2 case 2, $E[C_i^{S2C2}]$, is:

$$\begin{aligned} E[C_i^{S2C2}] &= \int_{a_i}^{T_i^e} TC_{S2C2a}(l_i) \frac{1}{b_i - a_i} dl_i + \int_{T_i^e}^{b_i} TC_{S2C2b}(l_i) \frac{1}{b_i - a_i} dl_i \\ &= + \frac{1}{12(a_i - b_i)} [-3a_i^3 d_i h + 3a_i^2 [d_i(2F + 48h - b_i h) + 2h(-Q_i + r_i)] \\ &+ 3a_i[(-1152 + b_i^2) d_i h + 4(k + FQ_i + 24hQ_i - Fr_i)] + b_i[12(288 d_i h - k \\ &- (F + 24h)(Q_i - r_i)) - 6b_i(d_i(F + 48h) + h(-Q_i + r_i)) + b_i^2 d_i(5h - 2s)] \\ &- 6((48 + b_i)hr_i - b_i^2 d_i s)T_i^e + 6(24 d_i h + hr_i - b_i d_i s)(T_i^e)^2 \\ &+ 2d_i(-h + s)(T_i^e)^3]. \end{aligned} \tag{10}$$

- Scenario 3: $b_i \leq T_i^e$

  The ordering costs in scenario 3 are also given by expression (7), as T_i^e occurs after the midpoint $(a_i + b_i)/2$.

  - Case 1: the petroleum is received before the midpoint of the time window ($l_i \leq (a_i+b_i)/2$)

    Figure 6(a) depicts the inventory level for this case, the shortage cost is 0 and the send-back cost is given by:

    $$\left[\left(\frac{a_i + b_i}{2}\right) d_i - l_i d_i\right] p,$$

    where $[(a_i+b_i)/2]d_i - l_i d_i$ is the difference between order quantity and the actual received quantity. The expected holding volumes before and after order arrival are

    $$r_i l_i - \frac{l_i^2 d_i}{2}$$

and

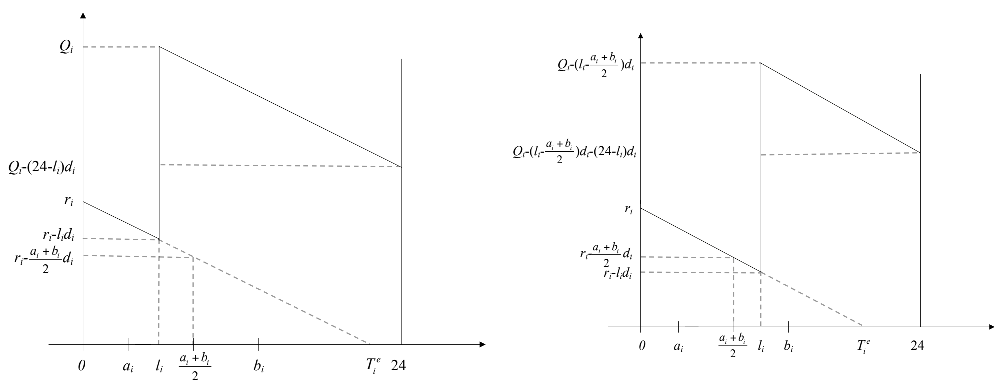
$$Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}.$$

Thus, the holding cost is

$$\left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + \left(Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right)\right] h.$$

We denote the total cost for the scenario 3 case 1 as $C_i^{S3C1}(l_i)$ and is thus given by

$$\begin{aligned} C_i^{S3C1}(l_i) = & \left[k + F\left[Q_i - \left[r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right]\right]\right] \\ & + \left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + \left(Q_i(24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right)\right] h \\ & + \left[\left(\frac{a_i + b_i}{2}\right) d_i - l_i d_i\right] p. \end{aligned} \tag{11}$$



(a) Case 1

(b) Case 2

Figure 6: The inventory model of scenario 3.

– Case 2: the petroleum is received after the midpoint of the time window ($l_i \geq (a_i+b_i)/2$))

Figure 6(b) depicts the inventory level for this case, there will be no shortage nor send-back cost and the holding cost is quite similar to the holding cost in case 1 except the expected holding volume after order arrival is

$$\left[Q_i - \left(l_i - \left(\frac{a_i + b_i}{2}\right) d_i\right)\right](24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}$$

Thus, the total cost for the scenario 3 case 2, denoted as $C_i^{S3C2}(l_i)$, is given by

$$\begin{aligned} C_i^{S3C2}(l_i) = & \left[k + F\left[Q_i - \left[r_i - \left(\frac{a_i + b_i}{2}\right) d_i\right]\right]\right] \\ & + \left[\left(r_i l_i - \frac{l_i^2 d_i}{2}\right) + \left[Q_i - \left(l - \left(\frac{a_i + b_i}{2}\right) d_i\right)\right](24 - l_i) - \frac{(24 - l_i)^2 d_i}{2}\right] h \end{aligned} \tag{12}$$

We then take an expectation over the limit for l_i from a_i to $(a_i+b_i)/2$ for case 1 and from $(a_i+b_i)/2$ to b_i for case 2 and the expected total cost of scenario 3, $E[TC_{S3}]$, is given by

$$
\begin{aligned}
E[C_i^{S3}] &= \int_{a_i}^{\frac{a_i+b_i}{2}} TC_{S3C1}(l_i)\frac{1}{b_i-a_i}dl_i + \int_{\frac{a_i+b_i}{2}}^{b_i} TC_{S3C2}(l_i)\frac{1}{b_i-a_i}dl_i \\
&= \frac{1}{48}[-17a_i^2 d_i h - 11b_i^2 d_i h - 48(288 d_i h - k - (F+24h)Q_i + Fr_i) \\
&+ 6b_i(d_i(4F+72h+p) + 4h(-Q_i+r_i)) + a_i(d_i(24F - 20(-36+b_i)h - 6p) \\
&+ 24h(-Q_i+r_i))]. \qquad (13)
\end{aligned}
$$

The objective of the inventory model is to select the optimal time window $[a_i, b_i]$ for each gas station among all possible discrete delivery times specified by transporters minimizing the total inventory costs. Based on the relationships among a_i, b_i, l_i, and T_i^e, we can identify which scenario will be applied and the expected total costs can be further calculated. The optimal time windows and order quantities for each gas station can be found by solving Equation (14).

$$
(q_i, [a_i, b_j]) \in \arg\min\{E[C_i(l_i)] = E[C_i^{S1}(l_i)] + E[C_i^{S2C1}(l_i)] + E[C_i^{S2C2}(l_i)] + E[C_i^{S3}(l_i)]\}. \quad (14)
$$

Furthermore, notice that for a given realization of l_i, it is possible to calculate the total cost incurred by the gas stations by selecting the scenario that corresponds to the value if l_i and evaluate the resulting value for $C_i(l_i)$, for all $i \in N$. We now discuss the distribution part of the problem.

5 The distribution problem of the transporter

The proposed integrated routing model seeks to determine distribution plan and the strategic decisions of the transporter by taking into account the customer decisions given by the inventory model.

5.1 Route generator

As mentioned before, we attempt to generate all the possible delivery routes for the gas stations that placed orders during each given day, based upon the results of the inventory models. For the cases where few gas stations are served per route, we generate all possible route permutations visiting one, two and three stations for a total of $O(|N|^3)$ routes. As for the cases where the route visits either four or five gas stations, since we solve both the truck loading model and the route scheduling model for all of those routes every day of the planning horizon, we do not attempt to generate all the total possibilities, as it would require solving two mixed-integer models for a total of $O(|N|^5)$ routes per day, which would take longer than the desired computational time limit. Notice that for each day of the planning horizon all routes must be tested. In other words, a route that was considered infeasible for the first day, may be feasible for the demands and time windows of the second day. Therefore, instead of generating all the possible permutations of four and five gas stations, we randomly generate a smaller subset of such routes to reduce the total computational load. Since, some good routes may not be generated in the random route generation, we test different sets of randomly generated routes to see the impact of the random generation.

Let R be the set of routes generated. For each of the routes in R, we solve the truck loading model that checks if the order quantities of the given route can be loaded into the truck compartments. Furthermore, if the route satisfies the truck loading constraints, we then generate the schedule for the route based on the given time windows of the customers of each route.

5.2 Truck loading model

In addition to satisfying the constraints given by the time windows, a feasible route must also satisfy the truck compartment capacity constraints. As the trucks used to deliver have multiple compartments with different capacities, the demand feasibility of the routes is checked in the truck loading model. If the demand of a route cannot be loaded into any truck, this route will be eliminated from the set of candidate routes. Therefore, for every possible route $r \in R$, we solve the following tuck loading model to test the loading feasibility of the route. The notation and the decision variables are described in Table 3.

Table 3: Mathematical notation for the truck loading model.

Notation	Description
N_r	The set of gas stations served by the route r being tested
C	The set of compartments for the given truck
q_i	The delivery quantity to station i
Q_c	The capacity of compartment c
y_{ic}	A binary variable equal to 1 if demand of gas station i is assigned to compartment c, and 0 otherwise

Then the truck loading model is given by:

$$q_i \leq \sum_{c \in C} Q_c y_{ic} \quad \forall i \in N_r, \tag{15}$$

$$\sum_{p \in P} y_{ic} \leq 1 \quad \forall c \in C, \tag{16}$$

$$y_{ic} \in \{0, 1\} \quad \forall i \in N_r, \, c \in C. \tag{17}$$

Constraints (15) enforce that the delivery quantity of station i cannot exceed the sum of compartment capacities of the loaded truck assigned to such a gas station. Constraints (16) ensure only one demand can be loaded into each compartment. Constraints (17) define the decision variables. Notice that the loading model is in fact a satisfiability problem, as any feasible solution can be used for loading the truck compartments. If route r does not satisfy the truck loading model, it is removed from set R.

Additionally, in this stage we compute the profit of the route, which is calculated by the revenue perceived by delivering the petroleum minus the travel costs of the route. Since the gas station visit sequence is given by the route, and the order quantities of each customer are calculated a priory by the inventory models, if there is a feasible loading distribution of route r—given by the solution of model (15)-(17)—we then calculate the corresponding profit of r and the transportation cost f the route and use it as an input for the truck routing model.

5.3 Route scheduling model

The route scheduling model aims to find the truck schedule of each candidate route $r \in R$, as well as a set of penalties for having late deliveries. Given the time windows obtained by the inventory models and the starting hour of the delivery shift, we check the feasibility of the time window constraints for each gas station of candidate route $r \in R$. Since the number of trucks is limited and the time window requirements for the gas stations of route r could be too close or potentially overlap, the delivery time windows are not always satisfied for all customers. For that reason, in the route scheduling model we assign penalties to the routes for which the stations receive petroleum at undesired times. The penalties consist of additional costs for truck arrival times either before a_i or after b_i, and are proportional to the truck arrival time at stations outside the specified time windows, as shown in (18). Notice that we also add additional penalties for deviating from the midpoint of the time windows. This is intended to ensure that, if possible, the tucks should try to arrive at the midpoint of the time windows to avoid the possibility for the customers to incur in shortage or send-back costs. Although, when considering the final value for the penalties of the route, only the penalties for deviating from the time window $[a_i, b_i]$ are considered. The routes with no deviations (positive or negative) are routes whose schedule guarantees that the truck arrives at every station at the desired time (the midpoint of time windows). The objective of this model is to find the optimal truck arrival time at each station in the route so that the penalties are minimized.

The definition of the notation used for this model is given in Table 4. The route scheduling model is formulated as follows:

$$\min \quad \sum_{i \in N_r} (u_i + v_i + \gamma m_i + \delta n_i) \tag{18}$$

$$\text{s.t.} \quad x_i - (a_i + b_i)/2 = v_i - u_i \quad \forall i \in N_r, \tag{19}$$

$$x_1 \geq h + t_0, \tag{20}$$

$$x_i \geq x_{i-1} + s_{t-1} + t_{i-1} \quad \forall i \in N_r \setminus \{1\}, \tag{21}$$

$$x_i + m_i \geq a_i \quad \forall i \in N_r, \tag{22}$$

$$x_i - n_i \leq b_i \quad \forall i \in N_r, \tag{23}$$

$$a_i \leq x_i \leq m \quad \forall i \in N_r, \tag{24}$$

$$x_i \in R^+, \quad u_i \in R^+, \quad v_i \in R^+, \quad m_i \in R^+, \quad n_i \in R^+ \quad \forall i \in N_r. \tag{25}$$

Table 4: Mathematical notation and decision variables for the route scheduling model.

Notation	Description
N_r	Subset of stations in given route r
h	The starting hour of the shift
m	The maximum delivery time in one day
s_i	The service time of station i
t_i	The trucks travel time from station i to station $i+1$, $\forall i \in N_r$
a_i	The earliest time window of station i
b_i	The latest time window of station i
γ	The penalty that truck arrival before a_i
δ	The penalty that truck arrival after b_i
x_i	Truck arrival time at station i
u_i	Negative truck arrival time deviation from the midpoint $(a_i + b_i)/2$ for station i
v_i	Positive truck arrival time deviation from the midpoint $(a_i + b_i)/2$ for station i
m_i	Amount of time the truck arrives before a_i for station i
n_i	Amount of time the truck arrives after b_i for station i

The objective function (18) minimizes penalties associated with the time deviations from the expected arrival times of route r. Constraints (19) define the positive and negative deviations of the delivery time from the midpoint of the time windows for each gas station i. Constraints (20) ensure that the truck arrival time at the first station of each route occurs after the starting hour of the shift plus the travel time from terminal to the first station. Constraints (21) enforce that the truck arrival time at all the gas stations in the route (except for the first station) is greater than the arrival time plus the service time of the preceding station, and travel time between the stations. Constraints (22)-(23) define the truck arrival time between the limits of the time window $[a_i, b_i]$. Constraints (24) require that the truck arrival time at station i lies within the earliest time window a_i and the maximum delivery time in one day m. Finally, constraints (25) define the decision variables.

Once the truck arrival time at each station of the route is known, one can compute the penalty for delivering petroleum to the gas stations in route r as:

$$\sum_{i \in N_r} (\gamma m_i + \delta n_i). \tag{26}$$

In addition to the penalties, the truck schedule can be recovered from the value of the $\mathbf{x}$'s variables.

5.4 ϵ-constraint method for the route scheduling problem

Typically, in the presence of time windows that overlap, or when some time windows are too narrow the time window requirements are difficult to satisfy. One of the possible situations that can occur is that a small set of the gas stations gets continuously penalized with late deliveries several days ov the time horizon, which may result in the potential loss of such customers. To avoid this, we attempt to balance the late deliveries over the time horizon by applying the ϵ-constraint method for the multiple time periods. For time period t, we first minimize the maximum time window violation of any gas station by transforming the model as follows:

$$
\begin{aligned}
\min \quad & z && (27)\\
\text{s.t.} \quad & z \geq u_i + v_i + \gamma m_i + \delta n_i + p^t_{cum} \quad \forall i \in N_r, && (28)\\
& (\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{n}) \in \Omega. && (29)
\end{aligned}
$$

Here p^t_{cum} represents the cumulative penalty of station i in route r for day t which is given by

$$
p^t_{cum} = \sum_{\tau=0}^{t-1} (\gamma m_i^\tau + \delta n_i^\tau) \tag{30}
$$

and Ω represents the constraint set given by (19)-(25). Also, variables $\mathbf{m}^\tau$ and $\mathbf{n}^\tau$ are assumed to be the optimal deviations of the scheduling problems of day τ. The solution found in the formulation above is denoted as z^*. We then minimize the summation of violations by replacing z by the optimal solution z^* in the formulation, which results in the following model.

$$
\begin{aligned}
\min \sum_{i \in N_r} & (u_i + v_i + \gamma m_i + \delta n_i) && (31)\\
\text{s.t.} \quad & z^* \geq u_i + v_i + \gamma m_i + \delta n_i + p^t_{cum} \quad \forall i \in N_r, && (32)\\
& (\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{n}) \in \Omega. && (33)
\end{aligned}
$$

5.5 Truck routing model

Finally, the objective of the truck routing model is to assign the optimal routes to the truck that will be used to fulfill the distribution plan. The trucks are allowed to make multiple deliveries provided that the delivery schedules of the routes do not overlap and the trucks return at the terminal by the allowable time limit. The notation and decision variable are defined in Table 5.

Table 5: Mathematical notation and decision variable for the truck routing model.

Notation	Description
K	The truck set
R_k	The possible route set of that can be served by truck k, $\forall k \in K$
N_r	The stations served by route r, $\forall r \in R_k$
a_{ir}	A binary parameter equal to 1 if station i is served by route r, $\forall r \in R_k$, $\forall k \in K$
ρ_r	The profit of route r, $\forall r \in R$
q_r	The penalty of route r if route r is a “late” route, $\forall r \in R$
x_{rk}	A binary variable that takes the value of 1 if route r is operated by truck k, $\forall r \in R_k$, $\forall k \in K$

Then the routing model is given by:

$$
\begin{aligned}
\max \quad & \sum_{r \in R_k} \sum_{k \in K} (\rho_r - q_r) x_{rk} && (34)\\
\text{s.t.} \quad & \sum_{r \in R_k} \sum_{k \in K} a_{ir} x_{rk} = 1 \quad \forall i \in N, && (35)
\end{aligned}
$$

$$\sum_{r \in N_k} x_{rk} \leq 1 \quad \forall t \in T,\, k \in K, \tag{36}$$

$$x_{rk} \in \{0, 1\} \quad \forall r \in R_k,\, k \in K. \tag{37}$$

For this formulation, the objective function (34) maximizes the total profit, which is difference between the profit and penalty of the routes. Constraints (35) state that each station is visited exactly once. Constraints (36) ensure that the delivery times of the selected routes cannot overlap for each discrete time period t. We consider T as 24 hours here. For example, the time slot between 2 and 3 can only be occupied by one route for each truck assuring that the selected routes will not overlap in the time slot from 2 to 3. Constraints (37) define the decision variables.

6 Computational results

The proposed solution framework was coded in Java 8 with the API of CPLEX 12.6. All the experiments were performed on an computer with an Intel(R) Xeon(R) E5645 CPU @ 2.40GHz processor and 32.0 GB of RAM. We first present the procedure for generating the test instances in subsection 6.1; The performance measures of the solution framework are presented in subsection 6.2; and then, the impact of several parameters on the generated distribution plans is studied thereafter.

6.1 Test problems

In order to evaluate the proposed petroleum distribution framework, we use a test bed of 15 randomly generated instances with 50 customers adapted from [26], after including the additional components introduced in this study. Among all the possible tank configurations provided in [26], we use as a base model a medium size tank configuration that we latter vary to analyze its impact on the solution quality (see Section 6.3). We converted the petroleum quantity units from liters to gallons and the distance units from kilometers to miles as per use in the typical context of U.S. scenarios. Furthermore, we introduced additional information required for the instances to be used in the context of our approach. The parameters used regarding the inventory costs and penalties can be found in Table 6. We consider a fleet of 24 trucks whose compartment compositions are given in Table 7. As for the demands, we generate demand rates for the test instances so that each station orders every one or two days. All this to see the patterns that emerge regarding the time windows and the impact of balancing late deliveries.

We further study the impact that the tank capacity of the gas stations and the number of randomly generated routes serving more than three gas stations have on the solutions; see Sections 6.3 and 6.4, respectively. We introduced a correlation between the demand rates and the tank capacities of the gas station to reflect the fact that gas stations with larger tanks are expected to have higher demand rates. The tank capacities we used are given in Table 8, labeled C1 to C5. Furthermore, we also vary the total number of randomly generated routes with more than three customers. We solve the replenishment problem for five scenarios labeled R1 to R5 in which we generate 100, 1000, 10000, 20000, and 50000 of such routes, respectively.

6.2 Performance of the solution framework

We tested the performance of the proposed approach over the 15 instances described in Section 6.1. We performed 50 different runs for each instance, varying the tank capacities, the demand rates,

Table 6: Parameters list.

Parameter	Value used
The unit order cost (F)	\$2.0
Fixed cost per delivery (k)	\$100
Holding cost(h)	\$0.2
Shortage cost(s)	\$0.4
Send-back cost(p)	\$0.5
Delivery time window limit hour	12
Average travel speed (mile/h)	40
Variable travel cost per mile	\$1.05
Service time (min)	45
Starting hour of the shift (hour)	3
Penalty of truck arrival time before a_i	2.0
Penalty of truck arrival time after b_i	2.0

Table 7: Truck configurations.

Type	Number of compartments	Capacities (gallons)	Number of trucks
1	6	4490, 1585, 2641, 2641, 1849, 2641	12
2	5	4226, 1585, 1585, 2641, 4226	6
3	4	4226, 2113, 3170, 3698	6

and the number of routes that are randomly generated. The data presented in Table 9 corresponds to the averages of the 50 runs and includes: for the gas stations, (1) the expected cost estimated by the inventory models, (2) the real cost obtained by calculating the costs based on the routes given by the distribution model and (3) the average deviation in hours of the delivery times with respect to the midpoints of the time windows per day. For the transporter, Table 9 presents the average profit and number of late deliveries per day. The real cost for the gas stations refers to the actual cost incurred by a station including all ordering cost, holding cost, shortage cost, and send-back cost after knowing when the station actually receive the petroleum by solving truck routing problem.

We observe that the real costs are lower than the expected costs in all instances. This is because the average tardiness for a gas station is less than one hour (the minimum expected length of the time windows). In contrast, the expected cost estimated by the inventory model considers possible scenarios where the petroleum is delivered significantly latter than the averages obtained by the proposed route scheduling model. Furthermore, we can see that even though 24 trucks are used to deliver all the demands, the number of late deliveries still ranges from 26.17 to 30.35 times everyday on average. The main reason is that most stations request similar and tight time windows, so it is difficult for the transporter to deliver all demands within the time windows even after using large fleet of trucks. Nevertheless, as mentioned before, the average late deliveries are not that far from the selected time windows.

Table 8: Tank capacities.

Scenarios	Tank 1	Tank 2	Tank 3
C1	5547	8395	9246
C2	5151	7795	8585
C3	4755	7196	7925
C4	4358	6596	7264
C5	3962	5996	6604

Table 9: The performance of the solution framework.

	Gas stations			Transporter	
Instance	Expected cost	Real cost	Tardiness	Profit	Number of late delivery
L1	29017.49	28368.67	0.75	3517.52	27.96
L2	28127.59	27554.23	0.72	3527.67	28.23
L3	27519.91	26929.52	0.72	3574.49	26.68
L4	28887.98	28285.76	0.72	3576.33	27.09
L5	26385.05	25743.66	0.86	2718.09	30.35
L6	27653.79	27077.03	0.77	3383.16	27.79
L7	28484.84	27850.98	0.77	3025.79	27.27
L8	28417.14	27799.55	0.74	3894.28	26.93
L9	28283.54	27732.59	0.67	3294.75	26.79
L10	27292.12	26722.60	0.77	2891.73	27.75
L11	27586.61	27011.34	0.78	3236.81	28.62
L12	28049.19	27515.62	0.71	3378.00	26.99
L13	28760.53	28175.04	0.70	3607.05	26.17
L14	28677.14	28128.30	0.71	3273.92	27.12
L15	28499.33	27859.18	0.78	3919.01	27.50

6.3 Impact of the tank capacity and demand rates

The tank capacities of the stations have a strong effect on the total order quantities because of the correlation that was introduced between the tank size and the demand rates. Consequently, there is also an effect on the total petroleum quantity distributed by the transporter. We ran our algorithms for solving the 15 instances using different levels of the tank capacities (C1-C5). Figure 7 depicts the average customer costs and the transporter profits for the different scenarios. Unsurprising, the higher the expected and real cost for the stations occur when the tank capacities are increased. From the station perspective, this is because the demands are larger which potentially increases the inventory costs.

When analyzing the late deliveries and the length of the tardiness periods, it is interesting to see a higher number of late deliveries and longer periods of tardiness when the tank capacities are reduced, see figures 8(a) and 8(b). With smaller tanks, the gas stations tend to place smaller orders more frequently. This in turn results in having trucks visiting more stations per route to cope with the increased number of order requests. In consequence, when more stations are visited by the same route, it is more likely that a larger number of late deliveries and longer periods of tardiness occur because the trucks need to finish several deliveries before visiting the last few stations in the

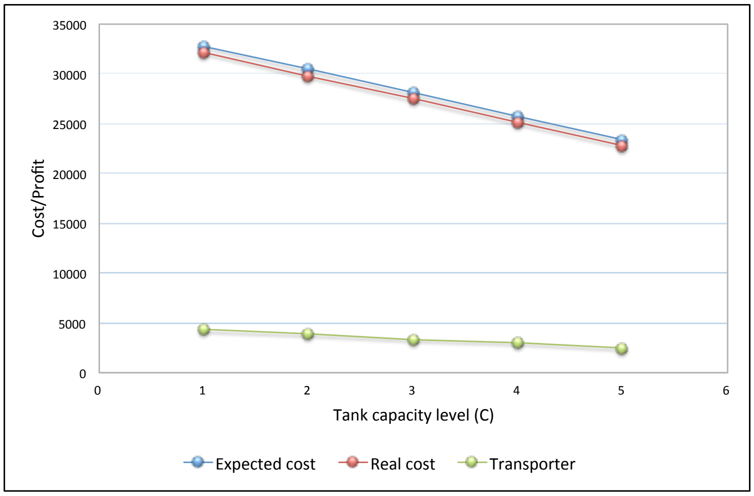
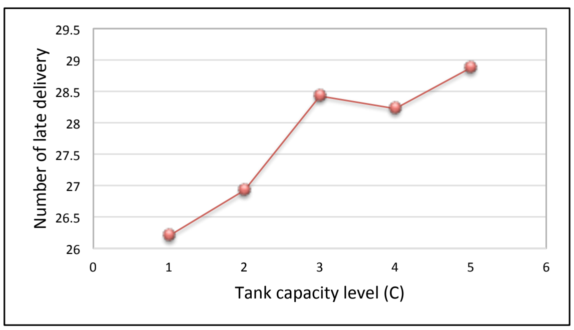
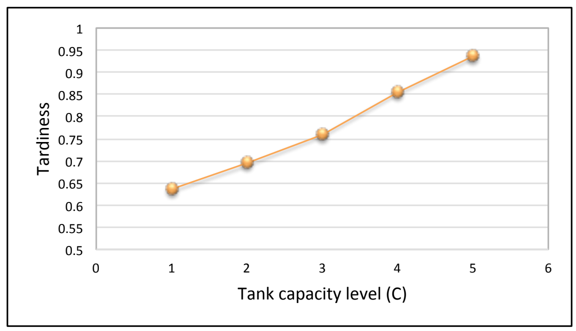


Figure 7: The economic impact of varying the tank capacity level.

route. This leads to longer waiting times for the rest of stations in such a route.



(a) The impact on the number of late deliveries



(b) The impact the length of the tradiness periods.

Figure 8: The delivery impact.

6.4 Impact of the number of randomly generated routes

In order to see the impact that using different numbers of randomly generated routes serving more than three gas stations has on the quality of the solution, we tested the five scenarios R1 to R5 in which we generate 100, 1000, 10000, 20000, and 50000 of such routes. We also vary the instance for the different tank capacity levels, using for testing cases C1 and C5. Table 10 shows that the percentage of profit improvement and the length of the tardiness periods do not improve significantly in higher level of tank capacity (C1). This is because the order quantities of the gas stations are relatively high. Thus, there is a low probability that trucks are able to load the demands of four stations or more in the same route. However, in the lower level of tank capacity (C5), the total order quantity may be low, so trucks have higher chance to visit more stations in the same route. The profit clearly increases with the use of more of these long routes. Note that the station tardiness increases by increasing the number of route generation in C5, but not in C1 due to the higher chance that trucks will visit up to four stations causing the late arrival time to the later stations in order within a route.

Table 10: The impact of the number of randomly generated route.

	C1				C5			
Instance	Profit	%Im	#LD	Tardiness	Profit	%Im	#LD	Tardiness
R1	4293.00	0.0000	26.47	0.63	2302.95	0.0000	28.61	0.75
R2	4294.90	0.0442	26.39	0.63	2338.10	1.5261	28.63	0.83
R3	4294.96	0.0014	26.42	0.64	2410.96	3.1163	29.27	0.93
R4	4299.11	0.0966	26.43	0.64	2449.00	1.5777	28.65	0.95
R5	4304.38	0.1226	26.20	0.64	2490.73	1.7042	28.89	0.94

%Im:= percentage of improvement; #LD:= number of late delivery.

6.5 The importance of balancing late deliveries among the customers

To evaluate the effect of balancing the late deliveries among the customers has on the total tardiness of the distribution plan, we use Instance L5—which is the one having the longest periods of tardiness, as presented in Table 9. We select the number of randomly generated routes to be 50000 (R5), to avoid the performance from affecting by choosing poor routes. Furthermore, we run the experiments for all the tank capacities C1 to C5. The results found when using our approach to attempt balancing late delivery are presented in Table 11. Additionally, we modify the optimization framework removing the ϵ-constraints approach introduced in Section 5.3, in an attempt to solve the problem without balancing the late deliveries. The results of this approach are presented in Table 12. In can be seen from these results that using the ϵ-constraints approach lowers the average tardiness and the difference between the maximum and minimum tardiness, which indicates the proposed policy tends to produce distribution schemes for which the overall tardiness is reduced.

Table 11: Tardiness when the late deliveries are balanced among the gas stations.

Instance	Avg	Max	Min	Max-Min
R5C1	0.71	1.99	0.25	1.74
R5C2	0.80	1.95	0.23	1.72
R5C3	0.88	2.52	0.16	2.36
R5C4	1.01	2.83	0.00	2.83
R5C5	1.05	2.83	0.00	2.83

Table 12: Tardiness when the late deliveries are not balanced among the gas stations.

Instance	Avg	Max	Min	Max-Min
R5C1	0.78	2.18	0.16	2.02
R5C2	0.86	2.06	0.23	1.82
R5C3	0.88	2.82	0.16	2.66
R5C4	1.05	3.21	0.12	3.09
R5C5	1.06	3.05	0.00	3.05

7 Concluding remarks

We have developed a methodological framework for designing the daily distribution and replenishment operations of petroleum products over a weekly horizon. The proposed models consider the option of allowing late deliveries in the cases where the expected time windows selected by the gas stations are too close to each other and potentially overlapping. The proposed approach consists of solving a series of optimization models for identifying the gas station petroleum demands and time windows, as well as for designing the distribution logistics. One of the main features of the optimization models is that the late deliveries are balanced among the customers in order to mitigate the overall customer dissatisfaction. The proposed approach was tested on a set of randomly generated problems adapted from the literature. The empirical results showed that the proposed approach is a viable option for designing distribution plans in contexts where the variability and complexity of the customer orders results in having unavoidable late deliveries. Further directions would be to embed the truck loading and the route scheduling models withing a column generation approach that, instead of testing a random sample of the possible routes, generates the optimal candidate routes sequentially, as needed.

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