MTH 131: Mathematical Analysis for Management, Fall 2017

Practice Midterm 3

Name: ______________________________

Student Number: ______________________________

Answer the questions in the spaces provided on the question sheets.
Show all of your work.
If you run out of room for an answer, continue on the back of the page.
You are allowed to use a non-graphing calculator.
You are allowed a one page formula sheet.

Page	Points	Score
3	7	
4	9	
5	8	
6	2	
7	8	
8	11	
9	10	
10	10	
11	10	
12	8	
13	9	
14	8	
Total:	100	

1. Evaluate the following indefinite integral: $\int 6x\,dx$ [2]

Full solution:

$$\int 6x\,dx = 6\int x^1\,dx = 6\frac{x^2}{2} + C = 3x^2 + C$$

$\int 6x\,dx =$ __**3x^2 + C**__ (Use C as the arbitrary constant.)

2. Find the following indefinite integral: $\int \frac{1}{x^6}\,dx$ [2]

Full solution:

$$\int \frac{1}{x^6}\,dx = \int x^{-6}\,dx = \frac{x^{-5}}{-5} + C = -\frac{1}{5x^5} + C$$

$\int x^{-6}\,dx =$ __**-1/(5x^5) + C**__ (Use C as the arbitrary constant.)

3. Evaluate: $\int \frac{8}{x}\,dx$ [2]

Full solution:

$$\int \frac{8}{x}\,dx = 8\int \frac{1}{x}\,dx = 8\ln|x| + C$$

$\int \frac{8}{x}\,dx =$ __**8 ln|x| + C**__ (Use C as the arbitrary constant.)

4. Discuss the validity of the following statement, and select the correct answer below. [1]

If n is an integer, then $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n.

Full solution:

This is a standard integration rule.

- ◯ This is a true statement for all n.
- ◯ This is a false statement for all n.
- √ **This is a true statement for all n except -1.**
- ◯ This is a true statement for all positive n.

5. Evaluate the following indefinite integral: $\int \frac{3}{\sqrt{x}}\,dx$ [3]

Full solution:

$$\int \frac{3}{\sqrt{x}}\,dx = 3\int x^{-1/2}\,dx = 3\frac{x^{1/2}}{1/2} + C = 6\sqrt{x} + C$$

$\int \frac{3}{\sqrt{x}}\,dx =$ **$6\sqrt{x}$ + C** (Use C as the arbitrary constant.)

6. Find the particular antiderivative of the following derivative that satisfies the given condition. [3]

$$\frac{dx}{dt} = 6e^t - 5; \qquad x(0) = 2$$

Full solution:

$$x = \int (6e^t - 5)\,dt = 6e^t - 5t + C$$
$$x(0) = 2 = 6e^0 - 5(0) + C \Rightarrow C = -4$$
$$\Rightarrow x(t) = 6e^t - 5t - 4$$

$x(t) =$ **6e^t - 5t - 4**

7. Find the indefinite integral: $\int \frac{x^3}{4+5x^4}\,dx$ [3]

Full solution:

Use the method of substitution: $u = 4 + 5x^4 \Rightarrow du = 20x^3\,dx \Rightarrow x^3\,dx = \frac{1}{20}du$. Then:

$$\int \frac{x^3}{4+5x^4}\,dx = \int \frac{1}{u}\frac{1}{20}\,du = \frac{1}{20}\int \frac{1}{u}\,du$$
$$= \frac{1}{20}\ln|u| + C = \frac{1}{20}\ln|4+5x^4| + C = \frac{\ln(4+5x^4)}{20} + C$$

Note that the absolute value can be dropped since $4 + 5x^4 > 0$ for all x.

$\int \frac{x^3}{4+5x^4}\,dx =$ **ln(4 + 5x^4)/20 + C** (Use C as the arbitrary constant.)

8. Find the indefinite integral: $\int x\sqrt{7-x^2}\,dx$ [3]

Full solution:

Use the method of substitution: $u = 7 - x^2 \Rightarrow du = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2}du$. Then:

$$\int x\sqrt{7-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\frac{u^{3/2}}{3/2} = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(7-x^2)^{3/2} + C$$

$\int x\sqrt{7-x^2}\,dx =$ **$-(7 - x^2)^{3/2}/3 + C$** (Use C as the arbitrary constant.)

9. Find the general solution for the first-order differential equation: $\frac{dy}{dx} = 28x$. [2]

Full solution:

Integrate both sides to find y as a function of x:

$$\frac{dy}{dx} = 28x \Rightarrow y = \int 28x\,dx = 28\frac{x^2}{2} + C = 14x^2 + C$$

$y =$ **$14x^2 + C$** (Use C as the arbitrary constant.)

10. Find the particular solution to the first-order differential equation that satisfies the given condition: [3]

$$\frac{dy}{dx} = e^{0.5x}; \qquad y(0) = 3$$

Full solution:

Find the general solution to the differential equation:

$$\frac{dy}{dx} = e^{0.5x} \Rightarrow y = \int e^{0.5x}\,dx = \frac{e^{0.5x}}{0.5} + C = 2e^{0.5x} + C$$

Use the given condition to find the particular solution:

$$y(0) = 3 = 2e^0 + C = 2 + C \Rightarrow C = 1$$

$y =$ **$2e^{0.5x} + 1$**

11. Give the order of the differential equation below, where y represents a function of x. [1]

$$y + x^4y'' - 2y' = 2$$

Full solution:

Find the highest order derivative in the left-hand side.

◯ The given equation is a first-order differential equation because it only involves a first derivative.

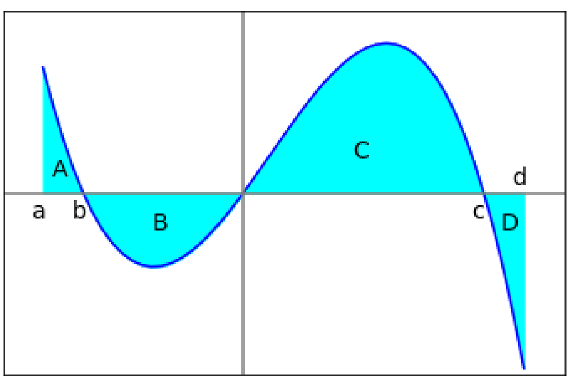
◯ The given equation is a first-order differential equation because it involves a first derivative but no higher derivative.

√ **The given equation is a second-order differential equation because it involves a second derivative but no higher derivative.**

◯ The given equation is a third-order differential equation because it involves a third derivative but no higher derivative.

12. Calculate the definite integral below by referring to the figure with the indicated areas. [1]

$$\int_0^c f(x)\,dx$$



Area A = 1.213 Area B = 2.71

Area C = 5.369 Area D = 1.611

Full solution:

The definite integral is the area under the graph of the function in the interval $[0, c]$. This is the area C.

$\int_0^c f(x)\,dx =$ **5.369** (Write the numerical value.)

13. Evaluate the integral: $\int_3^4 (2x+3)\,dx$. [3]

Full solution:

$$\int_3^4 (2x+3)\,dx = \left(x^2+3x\right)\Big|_3^4 = (4^2+3(4)) - (3^2+3(3)) = 28 - 18 = 10$$

$\int_3^4 (2x+3)\,dx =$ ______**10**______

14. Calculate the definite integral: $\int_3^6 \frac{7}{x}\,dx$. [3]

Full solution:

$$\int_3^6 \frac{7}{x}\,dx = 7\ln|x|\Big|_3^6 = 7(\ln 6 - \ln 3) = 7\ln 2$$

$\int_3^6 \frac{7}{x}\,dx =$ ______**7 ln 2**______

15. Evaluate the integral: $\int_4^4 \left(x^2-6x+9\right)^{15}\,dx$ [2]

Full solution:

The limits of integration are the same, so the result is zero.

$\int_4^4 \left(x^2-6x+9\right)^{15}\,dx =$ ______**0**______

16. Calculate the definite integral: $\int_0^1 5\sqrt[5]{x}\,dx$

Full solution:

$$\int_0^1 5\sqrt[5]{x}\,dx = 5\int_0^1 x^{1/5}\,dx = 5\left(\frac{x^{6/5}}{6/5}\right)\Big|_0^1 = 5\left(\frac{5}{6}\right)\left(1^{6/5}-0^{6/5}\right) = \frac{25}{6}$$

$\int_0^1 5\sqrt[5]{x}\,dx =$ ______**25/6**______

17. Find the area bounded by the graphs of the equations $f(x)$ and $g(x)$ over the given interval. [4]

$$f(x) = x^2 - 6; \quad g(x) = 3; \quad 0 \le x \le 3$$

Full solution:

$f(x) \le g(x)$ on [0, 3], so the area is:

$$A = \int_0^3 [g(x) - f(x)]\,dx = \int_0^3 3 - (x^2 - 6)\,dx = \int_0^3 (9 - x^2)\,dx$$
$$= \left(9x - \frac{x^3}{3}\right)\Big|_0^3 = \left(9(3) - \frac{3^3}{3}\right) - \left(9(0) - \frac{0^3}{3}\right) = 18$$

The area is ______**18**______ square units.

18. Find the area bounded by the graphs of the equations $f(x)$ and $g(x)$. [4]

$$f(x) = 4x^2; \quad g(x) = 36$$

Full solution:

The intersection points are where $f(x) = g(x)$:

$$4x^2 = 36 \Rightarrow x^2 = 9 \Rightarrow x = -3, 3$$

$f(x) \le g(x)$ on [-3, 3], so the area is:

$$A = \int_{-3}^3 [g(x) - f(x)]\,dx = \int_{-3}^3 \left(36 - 4x^2\right)\,dx$$
$$= \left(36x - \frac{4x^3}{3}\right)\Big|_{-3}^3 = \left(36(3) - \frac{4(3^3)}{3}\right) - \left(36(-3) - \frac{4(-3)^3)}{3}\right) = 144$$

The area, calculated to three decimal places, is ______**144**______ square units.

19. Find the Gini index of income concentration for the Lorenz curve given by: $f(x) = x^{2.4}$. [3]

Full solution:

$$GI = 2\int_0^1 [x - f(x)]\,dx = 2\int_0^1 [x - x^{2.4}]\,dx = 2\left(\frac{x^2}{2} - \frac{x^{3.4}}{3.4}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3.4}\right) \approx 0.412$$

The Gini index is ______**0.412**______ (Round to three decimal places.)

20. Find the total income produced by a continuous income stream in the first 3 years if the rate of flow is given by the following function, where t is time in years: [3]

$$f(t) = 400e^{0.05t}$$

Full solution:

$$\text{Total income } = \int_0^3 f(t)\,dt = \int_0^3 400e^{0.05t}\,dt = 400\frac{e^{0.05t}}{0.05}\Big|_0^3 = 8000(e^{0.15} - e^0) = 1294.67$$

The total income is **$1295** (Round to the nearest dollar.)

21. Find the future value at 3.25% interest, compounded continuously for 4 years, of the continuous income stream with rate of flow: [4]

$$f(t) = 1650e^{-0.02t}$$

Full solution:

$$FV = e^{rT}\int_0^T f(t)e^{-rt}\,dt = e^{0.0325(4)}\int_0^4 1650e^{-0.02t}e^{-0.0325t}\,dt = 1650e^{0.13}\int_0^4 e^{-0.0525t}\,dt$$

$$= 1650e^{0.13}\left(\frac{e^{-0.0525t}}{-0.0525}\right)\Bigg|_0^4 = 1650e^{0.13}\left(\frac{e^{-0.21} - e^0}{-0.0525}\right) = 6779.52$$

The future value is **$6780** (Round to the nearest dollar.)

22. Find the consumers' surplus at a price level of $\bar{p} = \$120$ for the price-demand equation below: [3]

$$p = D(x) = 400 - 0.02x$$

Full solution:

$$\bar{p} = \$120 = 400 - 0.02\bar{x} \Rightarrow \bar{x} = \frac{400 - 120}{0.02} = 14000$$

The consumers' surplus is then:

$$CS = \int_0^{\bar{x}} [D(x) - \bar{p}]\,dx = \int_0^{14000} [400 - 0.02x - 120]\,dx = \int_0^{14000} [280 - 0.02x]\,dx$$

$$= \left(280x - 0.01x^2\right)\Big|_0^{14000} = (280(14000) - 0.01(14000^2) = 1,960,000$$

The consumers' surplus is **$1,960,000** (Round to the nearest dollar.)

23. Find the producers' surplus at a price level of $\bar{p} = \$67$ for the price-supply equation below: [4]

$$p = S(x) = 10 + 0.1x + 0.0003x^2$$

Full solution:

$$\begin{aligned} &\bar{p} = 67 = 10 + 0.1\bar{x} + 0.0003\bar{x}^2 \Rightarrow 0.0003\bar{x}^2 + 0.1\bar{x} - 57 = 0 \\ &\Rightarrow 0 = 3x^2 + 1000x - 570000 = (x - 300)(3x + 1900) \\ &\Rightarrow \bar{x} = 300 \end{aligned}$$

The producer' surplus is then:

$$PS = \int_0^{\bar{x}} [\bar{p} - S(x)]\, dx = \int_0^{300} [67 - (10 + 0.1x + 0.0003x^2)]\, dx = \int_0^{300} [57 - 0.1x - 0.0003x^2]\, dx$$

$$= \left(57x - 0.05x^2 - 0.0001x^3\right)\Big|_0^{300} = 57(300) - 0.05(300^2) - 0.0001(300^3) = 9900$$

The producers' surplus is **\$9,900** (Round to the nearest dollar.)

24. Evaluate the integral: $\int xe^{6x}\, dx$ [6]

Full solution:

Integrating by parts, choose:

$$u = x \qquad dv = e^{6x}\, dx$$

$$du = dx \qquad v = \frac{e^{6x}}{6}$$

Then, by the integration-by-parts formula:

$$\int u\, dv = uv - \int v\, du = x\frac{e^{6x}}{6} - \int \frac{e^{6x}}{6}\, dx = \frac{xe^{6x}}{6} - \frac{e^{6x}}{36} + C$$

$\int xe^{6x}\, dx -$ **xe^{6x}/6 - e^{6x}/36 + C**

25. A company produces two models of a surfboard: a standard model and a competition model. The monthly cost function is given by: [1]

$$C(x, y) = 3000 + 150x + 310y$$

where x is the number of standard models produced per month and y is the number of competition models produced per month. Find $C(16, 8)$.

Full solution: $$C(16, 8) = 3000 + 150(16) + 310(8) = 7880$$

$C(16, 8) =$ **\$7,880**

26. The Cobb-Douglas production function for a bicycle company is given by: [2]

$$f(x, y) = 19x^{0.7}y^{0.3}$$

where x is the utilization of labor and y is the utilization of capital. If the company uses 1,211 units of labor and 1,742 units of capital, how many bicycle will be produced?

Full solution: $$f(1211, 1742) = 19(1211)^{0.7}(1742)^{0.3} = 25660.72$$

The company will produce **25,661** bicycles (Round to the nearest integer.)

27. Find $\dfrac{\partial z}{\partial y}$ if $z = x^2 - 4xy + 5y^2$ [3]

Full solution:
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 - 4xy + 5y^2) = \frac{\partial}{\partial y}x^2 - 4\frac{\partial}{\partial y}xy + 5\frac{\partial}{\partial y}y^2 = 0 - 4(x) + 5(2y) = -4x + 10y$$

$\dfrac{\partial z}{\partial y} =$ **-4x + 10y**

28. Find $f_{xx}(x, y)$ if $f(x, y) = (5x + y)^5$ [4]

Full solution:
$$f_x(x, y) = \frac{\partial}{\partial x}(5x + y)^5 = 5(5x + y)^4\frac{\partial}{\partial x}(5x + y) = 5(5x + y)^4(5) = 25(5x + y)^4$$
$$f_{xx}(x, y) = \frac{\partial}{\partial x}f_x(x, y) = \frac{\partial}{\partial x}25(5x + y)^4 = 25[4(5x + y)^3]\frac{\partial}{\partial x}(5x + y)$$
$$= 100(5x + y)^3(5) = 500(5x + y)^3$$

$f_{xx}(x, y) =$ **500(5x + y)**^{3}

29. Find the critical points of the function:

$$f(x, y) = x^2 - y^2 + 4x + 6y - 8$$

Use the second derivative test to determine if the critical points are local extrema.

(a) Find the critical points of $f(x, y)$. Use a comma to separate answers as needed. [4]

Full solution:

The critical points are where $f_x(x, y) = 0$ and $f_y(x, y) = 0$:

$$f_x(x, y) = 2x + 4 = 0 \Rightarrow x = -2, \qquad f_y(x, y) = -2y + 6 = 0 \Rightarrow y = 3$$

So, the function has a single critical point at (-2, 3)

The function $f(x, y)$ has critical points at **(-2, 3)**

(b) Use the second derivative test to determine if the critical points are local extrema. [1]

Full solution:

Applying the second-derivative test:

$$\begin{aligned} A &= f_{xx}(x, y) = 2 \\ B &= f_{xy}(x, y) = 0 \\ C &= f_{yy}(x, y) = -2 \\ AC - B^2 &= 2(-2) - 0^2 = -4 < 0 \text{ for all } (x, y) \end{aligned}$$

So, any critical points are saddle points.

Find the location of local minima.

○ The function has local minima at ________________

√ **The function has no local minima.**

(c) Find the location of local maxima. [1]

○ The function has local maxima at ________________

√ **The function has no local maxima.**

(d) Find the location of any saddle points. [1]

√ **The function has saddle points at (-2, 3)**

○ The function has no saddle points.

(e) Find the location of any critical points where the second-derivative test fails. [1]

○ The second-derivative test fails at ________________

√ **The second derivative test does not fail for any of the critical points.**

30. Explain why $f(x, y) = x^2$ has local extremum at infinitely many points. [1]

- ◯ $f_x(x, y)$ and $f_y(x, y)$ are always negative except at $x = 0$, where $f(x, y) = 0$, so f has the local maximum 0 at each point of the y-axis.
- √ **$f_x(x, y)$ and $f_y(x, y)$ are both equal to zero for $x = 0$ and y is any real number, meaning that each of these critical points represents a critical point. Since $f(x, y)$ is nonnegative and equals zero when $x = 0$, f has the local minimum 0 at each point of the y-axis.**
- ◯ $f_x(x, y)$ and $f_y(x, y)$ are both equal to zero for all points on the y-axis. Since $AC - B^2 < 0$ and $A < 0$ for all points on the x-axis, where $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $c = f_{yy}(a, b)$, and (a, b) represents a critical point, $f(x, y)$ has local minima for all points on the x-axis.
- ◯ $f_x(x, y)$ and $f_y(x, y)$ are always negativeexcept at $x = 0$, where $f(x, y) = 0$, so f has the local maximum 0 at each point of the x-axis.

31. Use the method of Lagrange Multipliers to maximize $f(x, y) = 2xy$ subject to $x + y = 6$ [8]

Full solution:

The constraint is $g(x, y) = x + y - 6 = 0$. Form the function:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = 2xy + \lambda(x + y - 6)$$

The critical points of this function are where:

$$\begin{aligned} F_x &= 2y + \lambda = 0 \Rightarrow \lambda = -2y \\ F_y &= 2x + \lambda = 0 \Rightarrow \lambda = -2x \\ F_\lambda &= x + y - 6 = 0 \end{aligned}$$

So, $-2y = -2x$, therefore $x = y$. Substituting into $F_\lambda = 0$ yields $2x - 6 = 0 \Rightarrow x = 3, y = 3$. The maximum value is then $f(3, 3) = 2(3)(3) = 18$.

The maximum value of $f(x, y)$ is **18** at $x =$ **3**, $y =$ **3**

32. Three pens of the same size are to be built along an existing fence. Let the fence parallel to the existing fence be of length x, and the length of the other four fences each be y. If 640 feet of fencing are available, what length should x and y be to produce the maximum total area? What is the maximum area? Use any technique you wish to find the maximum area. [8]

Full solution:

The total length of fence used is $x + 4y$. We want to maximize $A(x, y) = xy$ subject to the constraint $g(x, y) = x + 4y - 640 = 0$. Form the function:

$$F(x, y, \lambda) = A(x, y) + \lambda g(x, y) = xy + \lambda(x + 4y - 640)$$

The critical points of this function are where:

$$\begin{aligned} F_x &= y + \lambda = 0 \Rightarrow \lambda = -y \\ F_y &= x + 4\lambda = 0 \Rightarrow \lambda = -x/4 \\ F_\lambda &= x + 4y - 640 = 0 \end{aligned}$$

So, $-y = -x/4$, therefore $x = 4y$. Substituting into $F_\lambda = 0$ yields $4y + 4y - 640 = 0 \Rightarrow y = 80, x = 320$. The maximum value is then $A(320, 80) = 320(80) = 125600$.

The maximum area is ___**25,600**___ square feet when $x =$ ___**320**___ feet, $y =$ ___**80**___ feet.