MTH 131: Mathematical Analysis for Management, Fall 2017

Practice Midterm 1

Name: ______________________________

Student Number: ______________________________

Answer the questions in the spaces provided on the question sheets.

Show all of your working.

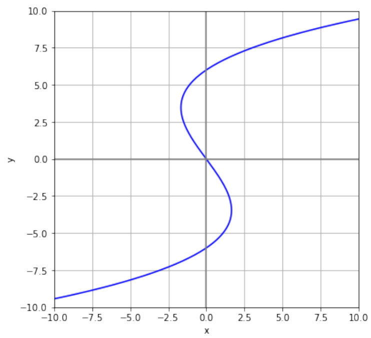
If you run out of room for an answer, continue on the back of the page.

You are allowed to use a non-graphing calculator.

You are allowed a half-page formula sheet.

Page	Points	Score
3	5	
4	9	
5	5	
6	15	
7	6	
8	6	
9	7	
10	4	
11	7	
12	2	
13	8	
14	7	
15	10	
16	9	
Total:	100	

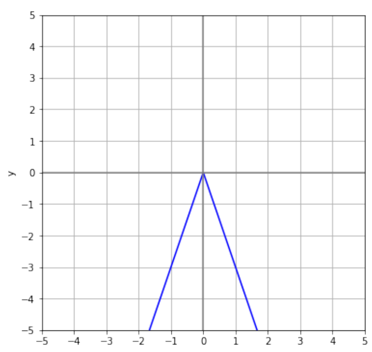
1. Does the following graph specify a function? [1]



$\surd$ **The graph does not specify a function.**

◯ The graph does specify a function.

2. The graph below involves a reflection in the x-axis and/or a vertical stretch or shrink of a basic function.



(a) Identify the basic function. [1]

◯ $\sqrt{x}$ ◯ x^3 $\surd$ $|x|$ ◯ x ◯ x^2 ◯ $\sqrt[3]{x}$

(b) Describe the transformation. [1]

◯ The graph is vertically stretched by 3.

$\surd$ **The graph is reflected about the x-axis and vertically stretched by 3.**

◯ The graph is vertically stretched by $\frac{1}{3}$.

◯ The graph is reflected about the x-axis and vertically stretched by $\frac{1}{3}$.

(c) Write an equation for the graph. [2]

Full solution:

The basic function $|x|$ is multiplied by -1 to reflect about the x-axis, then multiplied by 3 to vertically stretch it by 3.

$y =$ $-3|x|$

3. Find the domain of the function $F(x) = 9x^4 + 6x^2$. Write your answer in interval notation. [2]

Full solution:

$F(x)$ is a polynomial, so the domain is all real numbers.

The domain of the function is $\underline{\quad\quad\mathbf{(-\infty, \infty)}\quad\quad}$

4. Find and simplify each of the following for $f(x) = 5x - 3$.

(a) $f(x + h)$ [2]

Full solution:

$$f(x + h) = 5(x + h) - 3 = 5x + 5h - 3$$

$f(x + h) =$ **5x + 5h - 3**

(b) $f(x + h) - f(x)$ [2]

Full solution:

$$f(x + h) - f(x) = (5x + 5h - 3) - (5x - 3) = 5h$$

$f(x + h) - f(x) =$ **5h**

(c) $\dfrac{f(x + h) - f(x)}{h}$ [1]

Full solution:

$$\frac{f(x + h) - f(x)}{h} = \frac{5h}{h} = 5$$

$\dfrac{f(x + h) - f(x)}{h} =$ **5**

5. Use the revenue and cost functions below to answer the following questions.

$$R(x) = 80x - 3x^2 \qquad 1 \leq x \leq 20$$
$$C(x) = 130 + 15x \qquad 1 \leq x \leq 20$$

(a) What is the profit function $P(x)$? [2]

Full solution:

$$P(x) = R(x) - C(x) = (80x - 3x^2) - (130 + 15x) = -3x^2 + 65x - 130$$

$P(x) =$ **-3x^2 + 65x - 130**

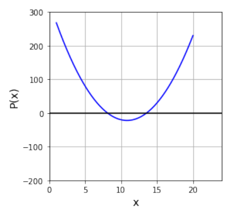
(b) What is the domain of the profit function? [1]

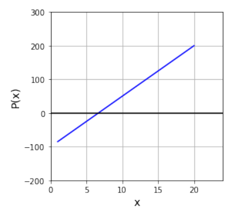
Full solution:
The domain of $P(x)$ is the interval where both $R(x)$ and $C(x)$ are defined. This is the intersection of the domains of $R(x)$ and $C(x)$.

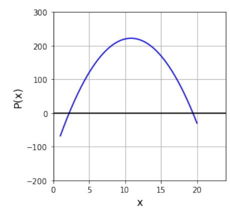
The domain is $\underline{\qquad 1 \leq x \leq 20 \qquad}$

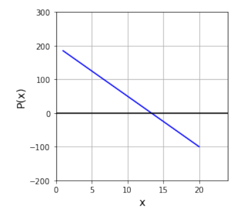
(c) Choose a possible graph for the profit function. [1]

Full solution:
$P(x)$ is a quadratic function with a negative leading coefficient, so the shape of the graph is a parabola opening downwards.

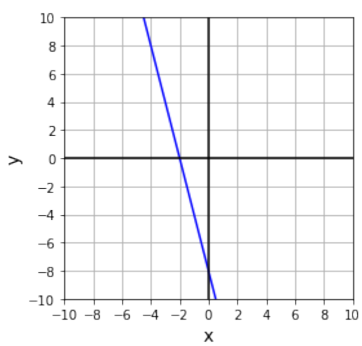
○ 

○ 

√ 

○ 

6. Use the following graph of a line to fill in the answers below.



(a) The x-intercept is $\underline{\qquad}$ **-2** $\underline{\qquad}$ [1]

Full solution:
The x-intercept is where the graph of the function intersects the x-axis.

(b) The y-intercept is $\underline{\qquad}$ **-8** $\underline{\qquad}$ [1]

Full solution:
The y-intercept is where the graph of the function intersects the y-axis.

(c) The slope is $\underline{\qquad}$ **-4** $\underline{\qquad}$ [1]

Full solution:
The slope is -8/2 = -4.

(d) The slope-intercept form of the equation of the line is **y = -4x - 8** [2]

Full solution:
The slope-intercept form of the equation of the line is $y = mx + b$.

(e) The standard form of the equation of the line is **4x + y = -8** [2]

Full solution:
The standard form of the equation of a line is $Ax + By = C$.

7. Consider the polynomial function $g(x) = x^2 + 6x + 5$.

(a) The degree of the polynomial is **2** [1]

Full solution:
The degree of the polynomial is the highest power of x.

(b) The y-intercept is **5** [1]

Full solution:
The y-intercept is given by $g(0) = 0^2 + 6(0) + 5$.

(c) The x-intercept(s) is/are **-1, -5** [2]

Full solution:
The x-intercept(s) are the solutions to $g(x) = 0 = (x + 5)(x + 1)$.

(d) The equation in vertex form is **g(x) = (x + 3)^2 - 4** [4]

Full solution:
$g(x) = x^2 + 6x + 5 = (x^2 + 6x + 9) - 4 = (x + 3)^2 - 4$

(e) The vertex is **(-3, -4)** [1]

Full solution:
The vertex form is $g(x) = a(x - h)^2 + k$, where the vertex is at (h, k).

(f) The function has a ◯ maximum √ **minimum** [1]

Full solution:
$g(x)$ is a quadratic function with a positive leading coefficient, so it has a minimum.

(g) The maximum or minimum value is **-4** [1]

Full solution:
The minimum value of $g(x) = (x+3)^2 - 4$ is when $(x+3)^2 = 0$ i.e. when $g(x) = -4$

(h) The range of the function is **[-4, ∞)** [1]

Full solution:

$g(x)$ has a minimum value of -4, but no maximum value.

8. A company is planning to manufacture snowboards. The fixed costs are \$100 per day and total costs are \$5900 per day at a daily output of 20 boards.

(a) Assuming that the total cost per day, $C(x)$, is linearly related to the total output per day, x, write an equation for the cost function. [2]

Full solution:

$C(x) =$ fixed costs $+$ variable costs $= a + bx = 100 + bx$.

When $x = 20$, $C(x) = 5900 = 100 + b(20)$, so $b = (5900 - 100)/20 = 290$

$C(x) =$ **290x + 100**

(b) The average cost per board for an output of x boards is given by $\overline{C}(x) = C(x)/x$. Find the average cost function. [1]

Full solution:

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{290x + 100}{x} = 290 + \frac{100}{x}$$

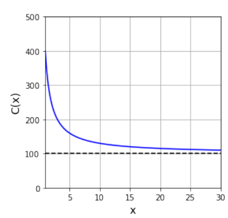
$\overline{C}(x) =$ **290 + 100/x**

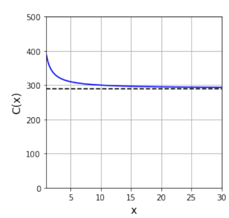
(c) One of the graphs below shows the average cost function, including asymptotes, for $1 \leq x \leq 30$. Choose the correct graph. [1]

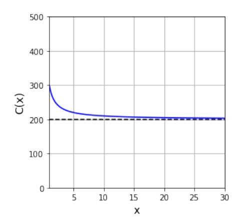
Full solution:

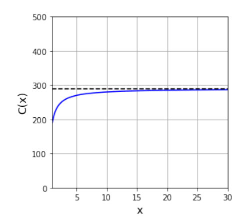
$$\overline{C}(x) = 290 + \frac{100}{x} > 290 \text{ for all } x$$

The second graph is the only one where $\overline{C}(x)$ stays above 290.

◯ 

√ 

◯ 

◯ 

(d) What does the average cost per board approach as production increases? [1]

Full solution:

The line $y = 290$ is a horizontal asymptote of the function $\overline{C}(x) = 290 + 100/x$, so this is the value that the average cost per board approaches.

The average cost per board approaches \$ **290**

9. Solve the given equation for x. Write your answer as a fraction or an integer [2]

$$g^{4-8x} = g^{3x-5}$$

Full solution:

$$\begin{aligned} g^{4-8x} &= g^{3x-5} \\ 4-8x &= 3x-5 \\ 9 &= 11x \\ x &= 9/11 \end{aligned}$$

$x =$ **9/11**

10. Write the expression $\log_6 1296 = 4$ in equivalent exponential form. [1]

Full solution:

$\log_b x = y$ is equivalent to $x = b^y$.

The equivalent exponential form is $\mathbf{1296 = 6^4}$

11. Write the equation in equivalent logarithmic form: [1]

$$4 = 16^{\frac{1}{2}}$$

Full solution:

$x = b^y$ is equivalent to $\log_b x = y$.

The equivalent logarithmic form is $\mathbf{\log_{16} 4 = 1/2}$

12. What are the domain and range of the function defined by $y = 1 + \ln(x-7)$? Write your answers in interval notation.

(a) What is the domain? [2]

Full solution:

The logarithm function $\ln x$ is defined for $x > 0$. So, the logarithm function $\ln(x-7)$ is defined for $x - 7 > 0$, or $x > 7$. Adding a 1 makes no difference to where the function is defined.

The domain is $\mathbf{(7, \infty)}$

(b) What is the range? [2]

Full solution:

The range of the logarithm function is $(-\infty, \infty)$. Subtracting 7 from x shifts the logarithm function right by 7 units, but does not change the range.

The range is **$(-\infty, \infty)$**

13. Write in terms of simpler forms: $\log_b M^9$ [1]

Full solution:

This is one of the "properties of logarithmic functions".

- ○ $M + \log_b 9$
- √ $9 \log_b M$
- ○ $9 + \log_b M$
- ○ $M \log_b 9$

14. How many years will it take \$6,000 to grow to \$9500 if it is invested at 3.75% compounded continuously? Round your answer to two decimal places. [4]

Full solution:

The formula for continuous compounding is $A = Pe^{rt}$. We have $A = 9500$, $P = 6000$ and $r = 0.0375$, and want to find t:

$$\begin{aligned} A &= Pe^{rt} \\ 9500 &= 6000e^{0.0375t} \\ \frac{9500}{6000} &= e^{0.0375t} \\ \ln\left(\frac{9500}{6000}\right) &= 0.0375t \\ t &= \frac{1}{0.0375}\ln\left(\frac{9500}{6000}\right) \approx 12.25 \end{aligned}$$

It will take **12.25** years.

15. Find the indicated quantity if it exists.

$$G(x) = \begin{cases} x^2 & \text{for } x < -1 \\ 3x & \text{for } x > -1 \end{cases}$$

(a) Select the correct choice below and fill in any answer boxes in your choice. [1]

Full solution:
For $x \to -1^+$, $x > -1$, so we use the formula $3x = 3(-1) = -3$

√ $\lim_{x\to -1^+} G(x) =$ **-3**

◯ The limit does not exist.

(b) Select the correct choice below and fill in any answer boxes in your choice. [1]

Full solution:
For $x \to -1^-$, $x < -1$, so we use the formula $x^2 = (-1)^2 = 1$

√ $\lim_{x\to -1^-} G(x) =$ **1**

◯ The limit does not exist.

(c) Select the correct choice below and fill in any answer boxes in your choice. [1]

Full solution:
The limit does not exist because the left-hand and right-hand limits are different.

◯ $\lim_{x\to -1} G(x) =$ ________

√ **The limit does not exist.**

16. If the statement below is always true, explain why. If not, give a counterexample. [1]

"If f is a function such that $\lim_{x\to 0} f(x)$ exists, then $f(0)$ exists."

Full solution:
By definition, the existence of a limit as $x \to c$ does not depend on the value of the function at c, or even if the function is defined at c.

◯ The statement is not always true. For example, if $f(x) = \dfrac{x}{x^2-1}$, then $\lim_{x\to 0} f(x) = 0$ but $f(0)$ does not exist.

√ **The statement is not always true. For example, if** $f(x) = \dfrac{x^2}{x}$**, then** $\lim_{x\to 0} f(x) = 0$ **but** $f(0)$ **does not exist.**

◯ The statement is always true. It is always the case that $\lim_{x\to c} f(x) = f(c)$.

◯ The statement is always true. Although it is possible for $f(0)$ to exist without $\lim_{x\to 0} f(x)$ existing, it is not possible for $\lim_{x\to 0} f(x)$ to exist without $f(0)$ also existing.

17. Consider the limit expression:

$$\lim_{x \to 9} \frac{x^2 - 4x - 45}{x - 9}$$

(a) Is the limit expression a $\frac{0}{0}$ indeterminate form? Choose the correct answer below. [1]

Full solution:

$$\lim_{x \to 9}(x^2 - 4x - 45) = 0 \text{ and } \lim_{x \to 9}(x - 9) = 0$$

So, the limit expression is a 0/0 indeterminate form.

√ **Yes**

◯ No

(b) Select the correct choice below and, if necessary, fill in the answer box with your choice. [2]

Full solution:

$$\lim_{x \to 9} \frac{x^2 - 4x - 45}{x - 9} = \lim_{x \to 9} \frac{(x+5)(x-9)}{x - 9} = \lim_{x \to 9}(x + 5) = 14$$

√ $\lim_{x \to 9} \frac{x^2 - 4x - 45}{x - 9} =$ **14**

◯ The limit does not exist and is neither ∞ nor $-\infty$.

18. Find the horizontal and vertical asymptotes for the function $f(x) = \frac{x^2 + 1}{x^2 - 1}$.

(a) Find the horizontal asymptote(s). Use a comma to separate answers as needed. [2]

Full solution:

The limit of a rational function at $\pm\infty$ depends only on the leading terms of the polynomials in the numerator and denominator:

$$\lim_{x \to -\infty} \frac{x^2}{x^2} = \lim_{x \to \infty} \frac{x^2}{x^2} = 1$$

√ $y =$ **1**

◯ There are no horizontal asymptotes.

(b) Find the vertical asymptote(s). Use a comma to separate answers as needed [2]

Full solution:

Vertical asymptotes occur in a rational function where the denominator is zero but the numerator is nonzero:

$$(x^2 - 1) = (x - 1)(x + 1) = 0$$

The numerator is nonzero for all x, so $x = -1$ and $x = 1$ are both vertical asymptotes.

√ $x =$ **-1, 1**

◯ There are no vertical asymptotes.

19. If the statement below is always true, explain why. If not, give a counterexample. [1]

"A polynomial function is continuous for all real numbers"

Full solution:

This was stated in class as a known property of polynomial functions.

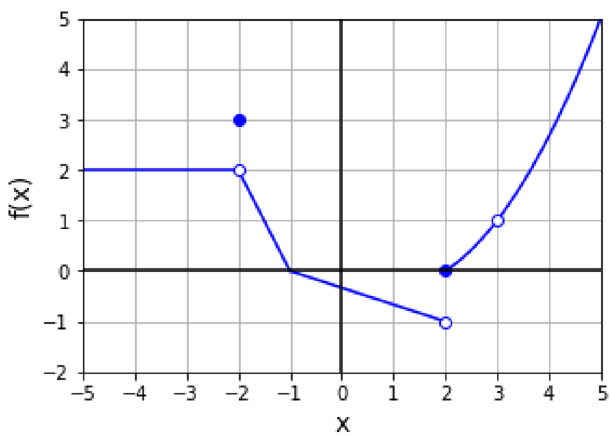
◯ The statement is false. A counterexample is $f(x) = 3x^2 - 2x + 1$.

◯ The statement is false. A counterexample is $f(x) = \dfrac{x^2-4}{x-2}$.

◯ The statement is false. A counterexample is $f(x) = \sqrt{x}$.

√ **The statement is true because, for any positive integer n, x^n is continuous for all real numbers.**

20. Use the graph of the function f shown to estimate the indicated quantities to the nearest integer. Select the correct choice in each case and, if necessary, fill in the answer box with your choice.



(a) Find the limit $\lim\limits_{x \to 2^-} f(x)$. [1]

Full solution:

Look at the value $f(x)$ approaches as x approaches 2 from the left.

√ $\lim\limits_{x \to 2^-} f(x) =$ **-1**

◯ The limit does not exist.

(b) Find the limit $\lim_{x \to 2^+} f(x)$. [1]

Full solution:

Look at the value $f(x)$ approaches as x approaches 2 from the right.

√ $\lim_{x \to 2^+} f(x) =$ ______**0**______

◯ The limit does not exist.

(c) Find the limit $\lim_{x \to 2} f(x)$. [1]

Full solution:

The left-hand and right hand limits are different, so the limit does not exist.

◯ $\lim_{x \to 2} f(x) =$ ____________

√ **The limit does not exist.**

(d) Find the function value $f(2)$. [1]

Full solution:

$f(2) = 0$, since there is a filled dot on the graph at the point (2, 0).

√ $f(2) =$ ______**0**______

◯ The value does not exist.

(e) Is f continuous at $x = 2$? [1]

Full solution:

f is not continuous at $x = 2$ since $\lim_{x \to 2} f(x)$ does not exist.

◯ Yes

√ **No**

21. Use the four-step process to find $r'(x)$ for $r(x) = 6 - 2x$. [4]

Full solution:

Use the four-step process to evaluate the limit $\lim_{h \to 0} \dfrac{r(x+h) - r(x)}{h}$.

$$\begin{aligned}
r(x+h) &= 6 - 2(x+h) = 6 - 2x - 2h \\
r(x+h) - r(x) &= (6 - 2x - 2h) - (6 - 2x) = -2h \\
\frac{r(x+h) - r(x)}{h} &= \frac{-2h}{h} \\
\lim_{h \to 0} \frac{r(x+h) - r(x)}{h} &= \lim_{h \to 0} \frac{-2h}{h} = \lim_{h \to 0} -2 = -2
\end{aligned}$$

$r'(x) =$ ______**-2**______

22. Determine whether f is differentiable at $x = 0$ by considering $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$: [4]

$$f(x) = 15 - |x|$$

Show all of your work, then choose the correct answer below.

Full solution:

Left-hand limit: $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{(15 - (-h)) - 15}{h} = 1$

Right-hand limit: $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{(15 - (h)) - 15}{h} = -1$

The limits need to be equal for the difference quotient limit to exist.

√ **The function f is not differentiable at $x = 0$ because the left- and right-hand limits of the difference quotient exist at $x = 0$, but are not equal.**

◯ The function f is differentiable at $x = 0$ because the graph has a sharp corner at $x = 0$.

◯ The function f is differentiable at $x = 0$ because both the left- and right-hand limits of the difference quotient exist at $x = 0$.

◯ The function f is not differentiable at $x = 0$ because the left- and right-hand limits of the difference quotient do not exist at $x = 0$.

23. Find $\frac{d}{dx} x^4$ [1]

Full solution:

By the Power Rule, $\frac{d}{dx} x^n = nx^{n-1}$

$\frac{d}{dx} x^4 =$ **$4x^3$**

24. Find $\frac{dy}{dx}$ for $y = \frac{1}{x^9}$. [2]

Full solution:

Write as $y = x^{-9}$, then apply the Power Rule.

$\frac{dy}{dx} =$ **$-9x^{-10}$, or $-9/x^{10}$**

25. Find $f'(t)$ if $f(t) = -7t^2 - 4t + 5$. [3]

Full solution:

$f(t)$ is a polynomial, so apply the Power Rule or Constant Function Rule to each term separately.

$f'(t) =$ **-14t - 4**

26. Find $G'(w)$ if $G(w) = \dfrac{7}{8w^4} + 9\sqrt{w}$. [4]

Full solution:

Rewrite as a sum of power functions: $G(w) = \dfrac{7}{8}w^{-4} + 9w^{1/2}$

The differentiate each term: $G'(w) = -\dfrac{28}{8}w^{-5} + \dfrac{9}{2}w^{-1/2}$

$G'(w) =$ **-7/(2w^5) + 9/(2$\sqrt{\text{w}}$)**

27. For $y = f(x) = 3x^5$, find the increments Δx and Δy, and find $\dfrac{\Delta y}{\Delta x}$, given $x_1 = 1$ and $x_2 = 2$.

(a) Find Δx [1]

Full solution:

$\Delta x = x_2 - x_1 = 2 - 1 = 1$

$\Delta x =$ **1**

(b) Find Δy [1]

Full solution:

$\Delta y = f(x_2) - f(x_1) = 3(2^5) - 3(1^5) = 3(32) - 3(1) = 93$

$\Delta y =$ **93**

(c) Find $\dfrac{\Delta y}{\Delta x}$ [1]

Full solution:

$\dfrac{\Delta y}{\Delta x} = \dfrac{93}{1} = 93$

$\dfrac{\Delta y}{\Delta x} =$ **93**

28. Find the differential dy [3]

$$y = 28 + 14x^4 - 2x^5$$

Full solution:

The formula for the differential is $dy = y'dx$

So: $dy = (56x^3 - 10x^4)dx$

$dy =$ **(56x^3 - 10x^4)dx**

29. Find the marginal revenue function. [3]

$$R(x) = x(22 - 0.08x)$$

Full solution:

Rewrite as power functions:

$$R(x) = 22x - 0.08x^2$$

The marginal revenue function is $R'(x) = 22 - 0.16x$.

$R'(x) =$ **22 - 0.16x**

30. Find the marginal profit function if cost and revenue are given by: [3]

$$C(x) = 231 + 0.8x \quad \text{and} \quad R(x) = 8x - 0.09x^2$$

Full solution:

The profit function is:

$$P(x) = R(x) - C(x) = (8x - 0.09x^2) - (231 + 0.8x) = -0.09x^2 + 7.2x - 231$$

The marginal profit function is:

$$P'(x) = -0.18x + 7.2$$

$P'(x) =$ **-0.18x + 7.2**