5 Integration

5.1 Antiderivatives and Indefinite Integrals

Antiderivatives

- A function F is an **antiderivative** of a function f if $F'(x) = f(x)$.
- If F and G are both antiderivatives of f, then F and G differ by a constant; that is, $F(x) = G(x) + k$ for some constant k.

Indefinite Integrals

- We use the symbol $\int f(x)\,dx$, called an **indefinite integral**, to represent the family of all antiderivatives of f, and we write

$$\int f(x)\,dx = F(x) + C$$

- The symbol $\int$ is called an **integral sign**, $f(x)$ is the **integrand**, and C is the **constant of integration**.

Properties of Indefinite Integrals

For k a constant:

- $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$
- $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)\,dx$

5.2 Integration by Substitution

Method of Substitution

- The **method of substitution** (also called the **change-of-variable method**) is a technique for finding indefinite integrals. It is based on the following formula, which is obtained by reversing the chain rule:

$$\int E'[I(x)]I'(x)\,dx = E[I(x)] + C$$

General Indefinite Integral Formulas

- $\displaystyle\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$
- $\displaystyle\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$
- $\displaystyle\int \frac{1}{f(x)} f'(x)\,dx = \ln[f(x)] + C$

Differentials

When using the method of substitution, it is helpful to use differentials as a bookkeeping device:

- The **differential** dx of the independent variable x is an arbitrary real number.
- The **differential** dy of the dependent variable y is defined by $dy = f'(x)dx$.

Guidelines for Using the Substitution Method

1. Select a substitution that appears to simplify the integrand. In particular, try to select u so that du is a factor in the integrand.
2. Express the integrand entirely in terms of u and du, completely eliminating the original variable and its differential.
3. Evaluate the new integral if possible.
4. Express the antiderivative found in step 3 in terms of the original variable.

5.3 Differential Equations; Growth and Decay

Differential Equation

- An equation is a **differential equation** if it involves an unknown function and one or more of the function's derivatives.
- An equation involving the first derivative of the unknown function but no second or higher-order derivatives is a **first-order** differential equation.

Slope Field

- A **slope-field** can be constructed for a differential equation by drawing a tangent line segment at each point (x, y) of a grid with slope given by the derivative at that point.
- The slope field gives a graphical representation of the functions that are solutions of the differential equation.

Exponential Growth Law

- The differential equation
$$\frac{dQ}{dt} = rQ$$
is called the **exponential growth law**.
- This equation describes a situation where the rate at which the unknown function Q increases is directly proportional to Q.
- The constant r is called the **relative growth rate**.
- The solutions of the exponential growth law are the functions
$$Q(t) = Q_0 e^{rt}$$
where Q_0 denotes $Q(0)$, the amount present at time $t = 0$.
- These functions can be used to solve problems in population growth and continuous compound interest.

5.4 The Definite Integral

Approximating Areas by Sums

- If the function f is positive on $[a, b]$, then the area between the graph of f and the x axis from $x = a$ to $x = b$ can be approximated by partitioning $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of equal length $\Delta x = (b - a)/n$ and summing the areas of n rectangles.
- **Left sum:** $L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x$
- **Right sum:** $R_n = \sum_{k=1}^{n} f(x_k)\Delta x$
- **Riemann sum:** $S_n = \sum_{k=1}^{n} f(c_k)\Delta x$, where each c_k belongs to the subinterval $[x_{k-1}, x_k]$.
- Left sums and right sums are special cases of Riemann sums in which c_k is the left endpoint and right endpoint, respectively, of the subinterval.

Approximation Error

- The **error in an approximation** is the absolute value of the difference between the approximation and the actual value.
- An **error bound** is a positive number such that the error is guaranteed to be less than or equal to that number.
- If $f(x) > 0$ and is either increasing on $[a, b]$ or decreasing on $[a, b]$, then

  $$|f(b) - f(a)| \cdot \frac{b - a}{n}$$

  is an error bound for the approximation of the area between the graph of f and the x axis, from $x = a$ to $x = b$, by L_n or R_n.

Limits of Approximation Sums

- If $f(x) > 0$ and is either increasing on $[a, b]$ or decreasing on $[a, b]$, then the left and right sums of $f(x)$ approach the same real number as $n \to \infty$.
- If f is a continuous function on $[a, b]$, then the Riemann sums for f on $[a, b]$ approach a real number limit I as $n \to \infty$.

The Definite Integral

- Let $f$be a continuous function of $[a, b]$. Then the limit I of Riemann sums for f on $[a, b]$ is called the **definite integral** of f from a yo b and is denoted

$$\int_a^b f(x)dx$$

- The **integrand** is $f(x)$, the **lower limit of integration** is a, and the **upper limit of integration** is b.
- Geometrically, the definite integral

$$\int_a^b f(x)dx$$

represents the cumulative sum of the signed areas between the graph of f and the x axis from $x = a$ to $x = b$.

Properties of the Definite Integral

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, k$ a constant
4. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
5. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

5.5 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If f is a continuous function on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(a) - F(b)$$

Evaluating Definite Integrals

- The fundamental theorem gives an easy and exact method for evaluating definite integrals, provided that we can find an antiderivative $F(x)$ of $f(x)$.
- In practice, we first find an antiderivative $F(x)$ using techniques for computing indefinite integrals, then calculate the difference $F(b) - F(a)$.
- If it is impossible to find an antiderivative, we must resort to left or right sums, or other approximation methods, to evaluate the definite integral.

Average Value

If f is a continuous function on $[a, b]$, then the **average value** of f over $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

6 Additional Integration Topics

6.1 Area Between Curves

Area Between Curves

If f and g are continuous and $f(x) \geq g(x)$ over the interval $[a, b]$, then the area bounded by $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is given exactly by

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Lorenz Curve

- A graphical representation of the distribution of income among a population can be found by plotting data points (x, y), where x **represents the cumulative percentage of families at or below a given income level** and y **represents the cumulative percentage of total family income received**.
- Regression analysis can be used to find a particular function $y = f(x)$, called a **Lorenz curve**, that best fits the data.

Gini Index

- A single number, the **Gini index**, measures income concentration:

$$\text{Gini index} = 2\int_0^1 [x - f(x)]\, dx$$

- A Gini index of 0 indicates **absolute equality**: all families share equally in the income.
- A Gini index of 1 indicates **absolute inequality**: one family has all of the income and the rest have none.

6.2 Applications in Business and Economics

Continuous Income Stream

- If the rate at which income is received - its **rate of flow** - is a continuous function $f(t)$ of time, then the income is said to be a **continuous income stream**.
- The **total income** produced by a continuous income stream from $t = a$ to $t = b$ is

$$\text{Total income } = \int_a^b f(t)\,dt$$

- The **future value** of a continuous income stream that is invested at rate r, compounded continuously, for $0 \le t \le T$, is

$$FV = e^{rT} \int_0^T f(t) e^{-rt}\,dt$$

Consumers' and Producers' Surplus

- If $(\overline{x}, \overline{p})$ is a point on the graph of a price-demand equation $p = D(x)$, then the **consumers' surplus** at a price level of $\overline{p}$ is

$$CS = \int_0^{\overline{x}} [D(x) - \overline{p}]\,dx$$

- The consumers' surplus represents the total savings to consumers who are willing to pay more than $\overline{p}$ but are still able to buy the product for $\overline{p}$.
- Similarly, for a point $(\overline{x}, \overline{p})$ on the graph of a price-supply equation $p = S(x)$, then the **producers' surplus** at a price level of $\overline{p}$ is

$$PS = \int_0^{\overline{x}} [\overline{p} - S(x)]\,dx$$

- The producers' surplus represents the total gain to producers who are willing to supply units at a lower price $\overline{p}$, but are still able to supply units at $\overline{p}$.

Equilibrium Price and Equilibrium Quantity

If $(\overline{x}, \overline{p})$ is the intersection point of a price-demand equation $p = D(x)$ and a price-supply equation $p = S(x)$, then $\overline{p}$ is called the **equilibrium price** and $\overline{x}$ is called the **equilibrium quantity**.

6.3 Integration By Parts

Integration-By-Parts Formula

Some indefinite integrals can be found by means of the **integration-by-parts formula**:

$$\int u\,dv = uv - \int v\,du$$

Select u and dv with the help of the following guidelines.

- The product $u\,dv$ must equal the original integrand.
- It must be possible to integrate dv (preferably by using standard formulas or simple substitutions).
- The new integral $\int v\,du$ should not be more complicated than the original integral $\int u\,dv$.
- For integrals involving $x^p e^{ax}$, try

$$u = x^p \quad \text{and} \quad dv = e^{ax}\,dx$$

- For integrals involving $x^p(\ln x)^q$, try

$$u = (\ln x)^q \quad \text{and} \quad dv = x^p\,dx$$

7 Multivariable Calculus

7.1 Functions of Several Variables

Functions of Two Independent Variables

- An equation of the form $z = f(x, y)$ describes a **function of two independent variables** if, for each permissible ordered pair (x, y), there is one and only one value of z determined by $f(x, y)$.
- The variables x and y are **independent variables**, and z is a **dependent variable**.
- The set of all ordered pairs of permissible values of x and y is the **domain** of the function.
- The set of all corresponding values $f(x, y)$ is the **range**.
- Functions of more than two independent variables are defined similarly.

The Graph of $z = f(x, y)$

- The graph of $z = f(x, y)$ consists of all triples (x, y, z) in **three-dimensional coordinate system** that satisfy the equation.
- The graph of the function $z = f(x, y) = x^2 + y^2$ is a **surface** with a local minimum.
- The graph of the function $z = g(x, y) = x^2 - y^2$ is a **surface** with a saddle point at (0, 0).

7.2 Partial Derivatives

First-Order Partial Derivatives

- If $z = f(x, y)$, then the **partial derivative of f with respect to x**, denoted as $\partial z/\partial x$, f_x, or $f_x(x, y)$, is:
$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
- Similarly the **partial derivative of f with respect to y**, denoted as $\partial z/\partial y$, f_y, or $f_y(x, y)$, is:
$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$
- The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are said to be **first-order partial derivatives**.

Second-Order Partial Derivatives

There are four **second-order partial derivatives** of $z = f(x, y)$:

- $f_{xx} = f_{xx}(x, y) = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$
- $f_{xy} = f_{xy}(x, y) = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$
- $f_{yx} = f_{yx}(x, y) = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$
- $f_{yy} = f_{yy}(x, y) = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$

7.3 Maxima and Minima

Local Maxima and Minima

- If $f(a, b) \geq f(x, y)$ for all (x, y) in a circular region in the domain of f with (a, b) as center, then $f(a, b)$ is a **local maximum**.
- If $f(a, b) \leq f(x, y)$ for all (x, y) in such a region, then $f(a, b)$ is a **local minimum**.
- If a function $f(x, y)$ has a local maximum or minimum at the point (a, b) and f_x and f_y exist at (a, b), then both first-order partial derivatives equal 0 at (a, b).

Second-Derivative Test for Local Extrema

If:

1. $z = f(x, y)$
2. $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [(a, b) is a critical point]
3. All second-order partial derivatives of f exist in some circular region containing (a, b) as center.
4. $A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b)$

Then:

1. If $AC - B^2 > 0$ and $A < 0$, $f(a, b)$ is a local maximum.
2. If $AC - B^2 > 0$ and $A > 0$, $f(a, b)$ is a local minimum.
3. If $AC - B^2 < 0$, f has a saddle point at (a, b).
4. If $AC - B^2 = 0$, the test fails.

7.4 Maxima and Minima Using Lagrange Multipliers

Method of Lagrange Multipliers

The **method of Lagrange multipliers** can be used to find local extrema of a function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$.

1. Write the problem in the form

$$\begin{array}{rl} \text{Maximize (or minimize)} & z = f(x, y) \\ \text{subject to} & g(x, y) = 0 \end{array}$$

2. Form the function F:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

3. Find the critical points of F; that is, solve the system

$$\begin{aligned} F_x(x, y, \lambda) &= 0 \\ F_y(x, y, \lambda) &= 0 \\ F_\lambda(x, y, \lambda) &= 0 \end{aligned}$$

4.
- If (x_0, y_0, λ_0) is the only critical point of F, we assume that (x_0, y_0) will always produce the solution to the problems we consider.
- If F has more than one critical point, we evaluate $z = f(x, y)$ at (x_0, y_0) for each critical point (x_0, y_0, λ_0) of F.
- For the problems we consider, we assume that the largest of these values is the maximum value of $f(x, y)$, subject to the constraint $g(x, y) = 0$, and the smallest is the minimum value of $f(x, y)$, subject to the constraint $g(x, y) = 0$.