5 Integration

5.1 Antiderivatives and Indefinite Integrals

Antiderivatives

• A function $F$ is an antiderivative of a function $f$ if $F'(x) = f(x)$.

• If $F$ and $G$ are both antiderivatives of $f$, then $F$ and $G$ differ by a constant; that is, $F(x) = G(x) + k$ for some constant $k$.

Indefinite Integrals

• We use the symbol $\int f(x) \, dx$, called an indefinite integral, to represent the family of all antiderivatives of $f$, and we write

$$\int f(x) \, dx = F(x) + C$$

• The symbol $\int$ is called an integral sign, $f(x)$ is the integrand, and $C$ is the constant of integration.

Properties of Indefinite Integrals

For $k$ a constant:

• $\int k f(x) \, dx = k \int f(x) \, dx$

• $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
5.2 Integration by Substitution

Method of Substitution

- The **method of substitution** (also called the **change-of-variable method**) is a technique for finding indefinite integrals. It is based on the following formula, which is obtained by reversing the chain rule:

\[
\int E'[I(x)]I'(x) \, dx = E[I(x)] + C
\]

General Indefinite Integral Formulas

- \[\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1\]
- \[\int e^{f(x)} f'(x) \, dx = e^{f(x)} + C\]
- \[\int \frac{1}{f(x)} f'(x) \, dx = \ln |f(x)| + C\]

Differentials

When using the method of substitution, it is helpful to use differentials as a bookkeeping device:

- The **differential** \( dx \) of the independent variable \( x \) is an arbitrary real number.
- The **differential** \( dy \) of the dependent variable \( y \) is defined by \( dy = f'(x) \, dx \).

Guidelines for Using the Substitution Method

1. Select a substitution that appears to simplify the integrand. In particular, try to select \( u \) so that \( du \) is a factor in the integrand.

2. Express the integrand entirely in terms of \( u \) and \( du \), completely eliminating the original variable and its differential.

3. Evaluate the new integral if possible.

4. Express the antiderivative found in step 3 in terms of the original variable.
5.3 Differential Equations, Growth and Decay

Differential Equation

- An equation is a **differential equation** if it involves an unknown function and one or more of the function’s derivatives.
- An equation involving the first derivative of the unknown function but no second or higher-order derivatives is a **first-order** differential equation.

Slope Field

- A **slope-field** can be constructed for a differential equation by drawing a tangent line segment at each point \((x, y)\) of a grid with slope given by the derivative at that point.
- The slope field gives a graphical representation of the functions that are solutions of the differential equation.

Exponential Growth Law

- The differential equation
  \[
  \frac{dQ}{dt} = rQ
  \]
  is called the **exponential growth law**.
- This equation describes a situation where the rate at which the unknown function \(Q\) increases is directly proportional to \(Q\).
- The constant \(r\) is called the **relative growth rate**.
- The solutions of the exponential growth law are the functions
  \[
  Q(t) = Q_0e^{rt}
  \]
  where \(Q_0\) denotes \(Q(0)\), the amount present at time \(t = 0\).
- These functions can be used to solve problems in population growth and continuous compound interest.
5.4 The Definite Integral

Approximating Areas by Sums

- If the function \( f \) is positive on \([a, b]\), then the area between the graph of \( f \) and the \( x \) axis from \( x = a \) to \( x = b \) can be approximated by partitioning \([a, b]\) into \( n \) subintervals \([x_{k-1}, x_k]\) of equal length \( \Delta x = (b - a)/n \) and summing the areas of \( n \) rectangles.

  - **Left sum:** \( L_n = \sum_{k=1}^{n} f(x_{k-1}) \Delta x \)
  
  - **Right sum:** \( R_n = \sum_{k=1}^{n} f(x_k) \Delta x \)
  
  - **Riemann sum:** \( S_n = \sum_{k=1}^{n} f(c_k) \Delta x \), where each \( c_k \) belongs to the subinterval \([x_{k-1}, x_k]\).

  - Left sums and right sums are special cases of Riemann sums in which \( c_k \) is the left endpoint and right endpoint, respectively, of the subinterval.

Approximation Error

- The **error in an approximation** is the absolute value of the difference between the approximation and the actual value.

- An **error bound** is a positive number such that the error is guaranteed to be less than or equal to that number.

- If \( f(x) > 0 \) and is either increasing on \([a, b]\) or decreasing on \([a, b]\), then

  \[
  |f(b) - f(a)| \cdot \frac{b - a}{n}
  \]

  is an error bound for the approximation of the area between the graph of \( f \) and the \( x \) axis, from \( x = a \) to \( x = b \), by \( L_n \) or \( R_n \).

Limits of Approximation Sums

- If \( f(x) > 0 \) and is either increasing on \([a, b]\) or decreasing on \([a, b]\), then the left and right sums of \( f(x) \) approach the same real number as \( n \to \infty \).

- If \( f \) is a continuous function on \([a, b]\), then the Riemann sums for \( f \) on \([a, b]\) approach a real number limit \( I \) as \( n \to \infty \).
The Definite Integral

- Let \( f \) be a continuous function of \( [a, b] \). Then the limit \( I \) of Riemann sums for \( f \) on \( [a, b] \) is called the **definite integral** of \( f \) from \( a \) to \( b \) and is denoted

\[
\int_a^b f(x) \, dx
\]

- The **integrand** is \( f(x) \), the **lower limit of integration** is \( a \), and the **upper limit of integration** is \( b \).

- Geometrically, the definite integral

\[
\int_a^b f(x) \, dx
\]

represents the cumulative sum of the signed areas between the graph of \( f \) and the \( x \) axis from \( x = a \) to \( x = b \).

Properties of the Definite Integral

1. \( \int_a^a f(x) \, dx = 0 \)
2. \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)
3. \( \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx, k \) a constant
4. \( \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \)
5. \( \int_c^a f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \)

5.5 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If \( f \) is a continuous function on \( [a, b] \) and \( F \) is any antiderivative of \( f \), then

\[
\int_a^b f(x) \, dx = F(a) - F(b)
\]
Evaluating Definite Integrals

• The fundamental theorem gives an easy and exact method for evaluating definite integrals, provided that we can find an antiderivative $F(x)$ of $f(x)$.

• In practice, we first find an antiderivative $F(x)$ using techniques for computing indefinite integrals, then calculate the difference $F(b) - F(a)$.

• If it is impossible to find an antiderivative, we must resort to left or right sums, or other approximation methods, to evaluate the definite integral.

Average Value

If $f$ is a continuous function on $[a, b]$, then the average value of $f$ over $[a, b]$ is defined to be

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$
6 Additional Integration Topics

6.1 Area Between Curves

Area Between Curves

If \( f \) and \( g \) are continuous and \( f(x) \geq g(x) \) over the interval \([a, b]\), then the area bounded by \( y = f(x) \) and \( y = g(x) \) for \( a \leq x \leq b \) is given exactly by

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

Lorenz Curve

- A graphical representation of the distribution of income among a population can be found by plotting data points \((x, y)\), where \( x \) represents the cumulative percentage of families at or below a given income level and \( y \) represents the cumulative percentage of total family income received.
- Regression analysis can be used to find a particular function \( y = f(x) \), called a Lorenz curve, that best fits the data.

Gini Index

- A single number, the Gini index, measures income concentration:

\[
\text{Gini index} = 2 \int_{0}^{1} [x - f(x)] \, dx
\]

- A Gini index of 0 indicates absolute equality: all families share equally in the income.
- A Gini index of 1 indicates absolute inequality: one family has all of the income and the rest have none.
6.2 Applications in Business and Economics

Probability Density Functions

- If any real number $x$ in an interval is a possible outcome of an experiment, then $x$ is said to be a **continuous random variable**.

- The probability distribution of a continuous random variable is described by a **probability density function** $f$ that satisfies the following conditions:
  1. $f(x) \geq 0$ for all real $x$.
  2. The area under the graph of $f(x)$ over the interval $(-\infty, \infty)$ is exactly 1.
  3. If $[c, d]$ is a subinterval of $(-\infty, \infty)$, then

\[
\text{Probability } (c \leq x \leq d) = \int_c^d f(x) \, dx
\]

Continuous Income Stream

- If the rate at which income is received - its **rate of flow** - is a continuous function $f(t)$ of time, then the income is said to be a **continuous income stream**.

- The **total income** produced by a continuous income stream from $t = a$ to $t = b$ is

\[
\text{Total income } = \int_a^b f(t) \, dt
\]

- The **future value** of a continuous income stream that is invested at rate $r$, compounded continuously, for $0 \leq t \leq T$, is

\[
FV = \int_0^T f(t)e^{r(T-t)} \, dt
\]
Consumers’ and Producers’ Surplus

• If \((x, p)\) is a point on the graph of a price-demand equation \(p = D(x)\), then the consumers’ surplus at a price level of \(p\) is
  \[
  CS = \int_0^x [D(x) - p] \, dx
  \]

• The consumers’ surplus represents the total savings to consumers who are willing to pay more than \(p\) but are still able to buy the product for \(p\).

• Similarly, for a point \((x, p)\) on the graph of a price-supply equation \(p = S(x)\), then the producers’ surplus at a price level of \(p\) is
  \[
  PS = \int_0^x [p - S(x)] \, dx
  \]

• The producers’ surplus represents the total gain to producers who are willing to supply units at a lower price \(p\), but are still able to supply units at \(p\).

Equilibrium Price and Equilibrium Quantity

If \((\bar{x}, \bar{p})\) is the intersection point of a price-demand equation \(p = D(x)\) and a price-supply equation \(p = S(x)\), then \(\bar{p}\) is called the equilibrium price and \(\bar{x}\) is called the equilibrium quantity.

6.3 Integration by Parts

Integration-by-Parts Formula

Some indefinite integrals can be found by means of the integration-by-parts formula:

\[
\int u \, dv = uv - \int v \, du
\]

Select \(u\) and \(dv\) with the help of the following guidelines.

• The product \(u \, dv\) must equal the original integrand.

• It must be possible to integrate \(dv\) (preferably by using standard formulas or simple substitutions).

• The new integral \(\int v \, du\) should not be more complicated than the original integral \(\int u \, dv\).

• For integrals involving \(x^p e^{ax}\), try
  \[
  u = x^p \quad \text{and} \quad dv = e^{ax} \, dx
  \]

• For integrals involving \(x^p (\ln x)^q\), try
  \[
  u = (\ln x)^q \quad \text{and} \quad dv = x^p \, dx
  \]
6.4 Other Integration Methods

Approximations to Definite Integrals

The trapezoidal rule and Simpson’s rule provide approximations of the definite integral that are more efficient than approximations by left or right sums, since fewer terms must be summed to achieve a given accuracy.

Trapezoidal Rule

- Let \( f \) be a function defined on an interval \([a, b]\).
- Partition \([a, b]\) into \(n\) subintervals of equal length \(\Delta x = (b - a)/n\) with endpoints
  \[ a = x_0 < x_1 < x_2 < \cdots < x_n = b \]
- Then
  \[ T_n = \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \Delta x/2 \]
  is an approximation of \( \int_a^b f(x) \, dx \).

Simpson’s Rule

- Let \( f \) be a function defined on an interval \([a, b]\).
- Partition \([a, b]\) into \(2n\) subintervals of equal length \(\Delta x = (b - a)/n\) with endpoints
  \[ a = x_0 < x_1 < x_2 < \cdots < x_{2n} = b \]
- Then
  \[ S_{2n} = \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{2n-1}) + f(x_{2n}) \right] \Delta x/3 \]
  is an approximation of \( \int_a^b f(x) \, dx \).

Tables of Integrals

A table of integrals is a list of integration formulas that can be used to find indefinite or definite integrals of frequently encountered functions.
7 Multivariable Calculus

7.1 Functions of Several Variables

Functions of Two Independent Variables

- An equation of the form \( z = f(x, y) \) describes a function of two independent variables if, for each permissible ordered pair \((x, y)\), there is one and only one value of \( z \) determined by \( f(x, y) \).
- The variables \( x \) and \( y \) are independent variables, and \( z \) is a dependent variable.
- The set of all ordered pairs of permissible values of \( x \) and \( y \) is the domain of the function.
- The set of all corresponding values \( f(x, y) \) is the range.
- Functions of more than two independent variables are defined similarly.

The Graph of \( z = f(x, y) \)

- The graph of \( z = f(x, y) \) consists of all triples \((x, y, z)\) in three-dimensional coordinate system that satisfy the equation.
- The graph of the function \( z = f(x, y) = x^2 + y^2 \) is a surface with a local minimum.
- The graph of the function \( z = g(x, y) = x^2 - y^2 \) is a surface with a saddle point at \((0, 0)\).

7.2 Partial Derivatives

First-Order Partial Derivatives

- If \( z = f(x, y) \), then the partial derivative of \( f \) with respect to \( x \), denoted as \( \frac{\partial z}{\partial x} \), \( f_x \), or \( f_x(x, y) \), is:
  \[
  \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
  \]
- Similarly the partial derivative of \( f \) with respect to \( y \), denoted as \( \frac{\partial z}{\partial y} \), \( f_y \), or \( f_y(x, y) \), is:
  \[
  \frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}
  \]
- The partial derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) are said to be first-order partial derivatives.
Second-Order Partial Derivatives

There are four second-order partial derivatives of \( z = f(x, y) \):

- \( f_{xx} = f_{xx}(x, y) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \)
- \( f_{xy} = f_{xy}(x, y) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \)
- \( f_{yx} = f_{yx}(x, y) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \)
- \( f_{yy} = f_{yy}(x, y) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \)

7.3 Maxima and Minima

Local Maxima and Minima

- If \( f(a, b) \geq f(x, y) \) for all \((x, y)\) in a circular region in the domain of \( f \) with \((a, b)\) as center, then \( f(a, b) \) is a local maximum.
- If \( f(a, b) \leq f(x, y) \) for all \((x, y)\) in such a region, then \( f(a, b) \) is a local minimum.
- If a function \( f(x, y) \) has a local maximum or minimum at the point \((a, b)\) and \( f_x \) and \( f_y \) exist at \((a, b)\), then both first-order partial derivatives equal 0 at \((a, b)\).

Second-Derivative Test for Local Extrema

If:

1. \( z = f(x, y) \)
2. \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \) \([a, b] \) is a critical point] 
3. All second-order partial derivatives of \( f \) exist in some circular region containing \((a, b)\) as center. 
4. \( A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b) \)

Then:

1. If \( AC - B^2 > 0 \) and \( A < 0 \), \( f(a, b) \) is a local maximum.
2. If \( AC - B^2 > 0 \) and \( A > 0 \), \( f(a, b) \) is a local minimum.
3. If \( AC - B^2 < 0 \), \( f \) has a saddle point at \((a, b)\).
4. If \( AC - B^2 = 0 \), the test fails.
7.4 Maxima and Minima Using Lagrange Multipliers

Method of Lagrange Multipliers

The **method of Lagrange multipliers** can be used to find local extrema of a function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$.

1. Write the problem in the form

   \[ \begin{array}{c}
   \text{Maximize (or minimize)} \\
   \text{subject to}
   \end{array} \]

   \[ z = f(x, y) \quad g(x, y) = 0 \]

2. Form the function $F$:

   \[ F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \]

3. Find the critical points of $F$; that is, solve the system

   \[ \begin{align*}
   F_x(x, y, \lambda) &= 0 \\
   F_y(x, y, \lambda) &= 0 \\
   F_\lambda(x, y, \lambda) &= 0
   \end{align*} \]

4. • If $(x_0, y_0, \lambda_0)$ is the only critical point of $F$, we assume that $(x_0, y_0)$ will always produce the solution to the problems we consider.

   • If $F$ has more than one critical point, we evaluate $z = f(x, y)$ at $(x_0, y_0)$ for each critical point $(x_0, y_0, \lambda_0)$ of $F$.

   • For the problems we consider, we assume that the largest of these values is the maximum value of $f(x, y)$, subject to the constraint $g(x, y) = 0$, and the smallest is the minimum value of $f(x, y)$, subject to the constraint $g(x, y) = 0$.

The method of Lagrange multipliers can be extended to functions with arbitrarily many independent variables with one or more constraints.