3 Additional Derivative Topics and Graphs

3.1 The Constant e and Continuous Compound Interest

The Constant e

The number e is defined as:

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{s\to 0}(1+s)^{1/s} = 2.718281828459\ldots$$

Continuous Compound Interest

If a principal P is invested at an annual rate r (expressed as a decimal) compounded continuously, then the amount A in the account at the end of t years is given by the **compound interest formula**:

$$A = Pe^{rt}$$

3.2 Derivatives of Exponential and Logarithmic Functions

Derivatives of Exponential Functions

For $b > 0$, $b \neq 1$:

- $\dfrac{d}{dx}e^x = e^x$
- $\dfrac{d}{dx}b^x = b^x \ln b$

Derivatives of Logarithmic Functions

For $b > 0$, $b \neq 1$, and $x > 0$:

- $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
- $\dfrac{d}{dx}\log_b x = \dfrac{1}{\ln b}\dfrac{1}{x}$

Change-of-Base Formulas

The **change-of-base formulas** allow conversion from base e to any base b, $b > 0$, $b \neq 1$:

- $b^x = e^{x \ln b}$
- $\log_b x = \dfrac{\ln x}{\ln b}$

3.3 Derivatives of Products and Quotients

Product Rule

If $y = f(x) = F(x)S(x)$, then

$$f'(x) = F(x)S'(x) + S(x)F'(x)$$

provided that both $F'(x)$ and $S'(x)$ exist.

Quotient Rule

If $y = f(x) = \dfrac{T(x)}{B(x)}$, then

$$f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2}$$

provided that both $T'(x)$ and $B'(x)$ exist.

3.4 The Chain Rule

Composite Functions

A function m is a **composite** of functions f and g if $m = f[g(x)]$.

Chain Rule

- The **chain rule** gives a formula for the derivative of the composite function $m(x) = E[I(x)]$:
$$m'(x) = E'[I(x)]I'(x)$$
- A special case of the chain rule is called the **general power rule**:
$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$
- Other special cases of the chain rule are the following **general derivative rules**:
$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)}f'(x) \quad \text{and} \quad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

3.5 Implicit Differentiation

Implicit Differentiation

- If $y = y(x)$ is a function defined implicitly by the equation $F(x, y) = 0$, then we use **implicit differentiation** to find an equation in x, y, and y'.
- Implicit differentiation involves taking the derivative of every term in the equation $F(x, y) = 0$ with respect to x, then solving for y'.
- Finding the derivative of terms involving y usually involve the chain rule, where y is considered as a function of x.
- The answer will generally involve both x and y since we need to know which point on the graph of $F(x, y) = 0$ we are finding the tangent line to.

3.6 Related Rates

Related-Rates Problems

- If x and y represent quantities that are changing with respect to time and are related by the equation $F(x, y) = 0$, then implicit differentiation produces an equation that relates x, y, dy/dt and dx/dt.
- Problems of this type are called **related-rates problems**.
- Implicit differentiation for related rates problems involves taking the derivative of every term in the equation $F(x, y) = 0$ with respect to t, where both x and y are considered as functions of t. Finding the derivative of terms involving x and y will usually involve the chain rule.

3.1 Elasticity of Demand

Relative Rates of Change

- The **relative rate of change**, or the **logarithmic derivative**, of a function $f(x)$ is $f'(x)/f(x)$.
- The **percentage rate of change** is $100 \times [f'(x)/f(x)]$.

Elasticity of Demand

- If price and demand are related by $x = f(p)$, then the **elasticity of demand** is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}}$$

- If $R(p) = pf(p)$ is the revenue function, then $R'(p)$ and $[1 - E(p)]$ always have the same sign.
- **Demand is inelastic** if $0 < E(p) < 1$. Demand is not sensitive to changes in price; a percentage change in price produces a smaller percentage change in demand. A price increase will increase revenue.
- **Demand is elastic** if $E(p) > 1$; a percentage change in price produces a larger percentage change in demand. A price increase will decrease revenue.
- **Demand has unit elasticity** if $E(p) = 1$; a percentage change in price produces the same percentage change in demand.

4 Graphing and Optimization

4.1 First Derivative and Graphs

Increasing and Decreasing Functions

- A function f is **increasing** on an interval (a, b) if $f(x_2) > f(x_1)$ whenever $a < x_1 < x_2 < b$.
- A function f is **decreasing** on an interval (a, b) if $f(x_2) < f(x_1)$ whenever $a < x_1 < x_2 < b$.
- For the interval (a, b), if $f' > 0$, then f is increasing, and if $f' < 0$, then f is decreasing.
- A sign chart for f' can be used to tell where f is increasing or decreasing.

Critical Numbers

- A real number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist is called a **critical number** of f.
- So, a critical number for f is a partition number for f' that also belongs in the domain of f.

Local Extrema

- A value $f(c)$ is a **local maximum** if there is an interval (m, n) containing c such that $f(x) \leq f(c)$ for all x in (m, n).
- A value $f(c)$ is a **local minimum** if there is an interval (m, n) containing c such that $f(x) \geq f(c)$ for all x in (m, n).
- A local maximum or local minimum is called a **local extremum**.
- If $f(c)$ is a local extremum, then c is a critical number of f.
- The **first-derivative test for local extrema** identifies local maxima and minima of f by means of a sign chart for f'.

4.2 Second Derivative and Graphs

Concavity

- The graph of f is **concave upward** on (a, b) if f' is increasing on (a, b) and is **concave downward** if f' is decreasing on (a, b).
- For the interval (a, b), if $f'' > 0$ then f is concave upward, and if $f'' < 0$, then f is concave downward.
- A sign chart for f'' can be used to tell where f is concave upward or concave downward.
- An **inflection point** of f is a point $(c, f(c))$ on the graph of f where the concavity changes.

4.3 L'Hôpital's Rule

0/0 Indeterminate Forms

- L'Hôpital's rule for 0/0 indeterminate forms states: if $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then

  $$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

  provided the second limit exists or is ∞ or $-\infty$.
- Always check to make sure that L'Hôpital's rule is applicable before using it.
- L'Hôpital's rule remains valid if the symbol $x \to c$ is replaced everywhere it occurs by one of

  $$x \to c^+ \qquad x \to c^- \qquad x \to \infty \qquad x \to -\infty$$

∞/∞ Indeterminate Forms

L'Hôpital's rule is also valid for indeterminate forms $\dfrac{\pm\infty}{\pm\infty}$.

4.4 Curve-Sketching Techniques

Graphing Strategy

1. *Analyze $f(x)$.*
   - Find the domain of f.
   - Find the intercepts.
   - Find asymptotes.
2. *Analyze $f'(x)$.*
   - Find the partition numbers for f' and the critical numbers of f.
   - Construct a sign chart for $f'(x)$.
   - Determine the intervals on which f is increasing and decreasing.
   - Find the local maxima and minima of f.
3. *Analyze $f''(x)$.*
   - Find the partition numbers for $f''(x)$.
   - Construct a sign chart for $f''(x)$.
   - Determine the intervals on which the graph of f is concave upward and concave downward.
   - Find the inflection points of f.
4. *Sketch the graph of f.*
   - Draw asymptotes and locate intercepts, local maxima and minima, and inflection points.
   - Sketch in what you know from steps 1-3.
   - Plot additional points as needed and complete the sketch.

4.5 Absolute Maxima and Minima

Absolute Extrema

- If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f.
- If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f.
- An absolute maximum or absolute minimum is called an **absolute extremum**.
- Absolute extrema, if they exist, must occur at critical numbers or endpoints.
- To find the absolute maximum and absolute minimum of a continuous function f on a closed interval, identify the endpoints and critical numbers in the interval, evaluate the function f at each of them, and choose the largest and smallest values of f.

Second-Derivative Test for Local Extrema

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.
- No conclusion can be drawn if $f''(c) = 0$.

Second-Derivative Test for Absolute Extrema on an Open Interval

This is applicable when a function f is continuous on an open interval I and there is only one critical number c in I:

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.

4.6 Optimization

Strategy for Solving Optimization Problems

1. Introduce variables, look for relationships among the variables, and construct a mathematical model of the form

   Maximize (or minimize) $f(x)$ on the interval I.

2. Find the critical numbers of $f(x)$.
3. Use the procedures developed in Section 4.5 to find the absolute maximum (or minimum) of $f(x)$ on the interval I and the numbers x where this occurs.
4. Use the solution to the mathematical model to answer all the questions asked in the problem.