3 Additional Derivative Topics and Graphs

3.1 The Constant $e$ and Continuous Compound Interest

**The Constant $e$**

The number $e$ is defined as:

$$\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{s \to 0} (1 + s)^{1/s} = 2.718281828459 \ldots$$

**Continuous Compound Interest**

If a principal $P$ is invested at an annual rate $r$ (expressed as a decimal) compounded continuously, then the amount $A$ in the account at the end of $t$ years is given by the **compound interest formula**:

$$A = Pe^{rt}$$

3.2 Derivatives of Exponential and Logarithmic Functions

**Derivatives of Exponential Functions**

For $b > 0$, $b \neq 1$:

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} b^x = b^x \ln b$

**Derivatives of Logarithmic Functions**

For $b > 0$, $b \neq 1$, and $x > 0$:

- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $\frac{d}{dx} \log_b x = \frac{1}{\ln b} \frac{1}{x}$
Change-of-Base Formulas

The change-of-base formulas allow conversion from base $e$ to any base $b$, $b > 0$, $b \neq 1$:

- $b^x = e^{x \ln b}$
- $\log_b x = \frac{\ln x}{\ln b}$

3.3 Derivatives of Products and Quotients

Product Rule

If $y = f(x) = F(x)S(x)$, then

$$f'(x) = F(x)S'(x) + S(x)F'(x)$$

provided that both $F'(x)$ and $S'(x)$ exist.

Quotient Rule

If $y = f(x) = \frac{T(x)}{B(x)}$, then

$$f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2}$$

provided that both $T'(x)$ and $B'(x)$ exist.

3.4 The Chain Rule

Composite Functions

A function $m$ is a composite of functions $f$ and $g$ if $m = f[g(x)]$. 
Chain Rule

- The chain rule gives a formula for the derivative of the composite function \( m(x) = E[I(x)] \):
  \[
m'(x) = E'[I(x)]I'(x)
  \]

- A special case of the chain rule is called the general power rule:
  \[
  \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)
  \]

- Other special cases of the chain rule are the following general derivative rules:
  \[
  \frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)}f'(x) \quad \text{and} \quad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)
  \]

3.5 Implicit Differentiation

Implicit Differentiation

If \( y = y(x) \) is a function defined implicitly by the equation \( F(x, y) = 0 \), then we use implicit differentiation to find an equation in \( x, y, \) and \( y' \).

3.6 Related Rates

Related-Rates Problems

- If \( x \) and \( y \) represent quantities that are changing with respect to time and are related by the equation \( F(x, y) = 0 \), then implicit differentiation produces an equation that relates \( x, y, \) dy/dt and dx/dt.

- Problems of this type are called related-rates problems.

3.7 Elasticity of Demand

Relative Rates of Change

- The relative rate of change, or the logarithmic derivative, of a function \( f(x) \) is \( f'(x)/f(x) \).

- The percentage rate of change is \( 100 \times [f'(x)/f(x)] \).
Elasticity of Demand

- If price and demand are related by \( x = f(p) \), then the **elasticity of demand** is given by
  \[
  E(p) = -\frac{p f'(p)}{f(p)} = -\frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}}
  \]

- If \( R(p) = pf(p) \) is the revenue function, then \( R'(p) \) and \( [1 - E(p)] \) always have the same sign.

- **Demand is inelastic** if \( 0 < E(p) < 1 \). Demand is not sensitive to changes in price; a percentage change in price produces a smaller percentage change in demand. A price increase will increase revenue.

- **Demand is elastic** if \( E(p) > 1 \); a percentage change in price produces a larger percentage change in demand. A price increase will decrease revenue.

- **Demand has unit elasticity** if \( E(p) = 1 \); a percentage change in price produces the same percentage change in demand.
4 Graphing and Optimization

4.1 First Derivative and Graphs

Increasing and Decreasing Functions

- A function \( f \) is **increasing** on an interval \((a, b)\) if \( f(x_2) > f(x_1) \) whenever \( a < x_1 < x_2 < b \).
- A function \( f \) is **decreasing** on an interval \((a, b)\) if \( f(x_2) < f(x_1) \) whenever \( a < x_1 < x_2 < b \).
- For the interval \((a, b)\), if \( f' > 0 \), then \( f \) is increasing, and if \( f' < 0 \), then \( f \) is decreasing.
- A sign chart for \( f' \) can be used to tell where \( f \) is increasing or decreasing.

Critical Numbers

- A real number \( x \) in the domain of \( f \) such that \( f'(x) = 0 \) or \( f'(x) \) does not exist is called a **critical number** of \( f \).
- So, a critical number for \( f \) is a partition number for \( f' \) that also belongs in the domain of \( f \).

Local Extrema

- A value \( f(c) \) is a **local maximum** if there is an interval \((m, n)\) containing \( c \) such that \( f(x) \leq f(c) \) for all \( x \) in \((m, n)\).
- A value \( f(c) \) is a **local minimum** if there is an interval \((m, n)\) containing \( c \) such that \( f(x) \geq f(c) \) for all \( x \) in \((m, n)\).
- A local maximum or local minimum is called a **local extremum**.
- If \( f(c) \) is a local extremum, then \( c \) is a critical number of \( f \).
- The **first-derivative test for local extrema** identifies local maxima and minima of \( f \) by means of a sign chart for \( f' \).
4.2 Second Derivative and Graphs

Concavity

- The graph of \( f \) is **concave upward** on \((a, b)\) if \( f' \) is increasing on \((a, b)\) and is **concave downward** if \( f' \) is decreasing on \((a, b)\).
- For the interval \((a, b)\), if \( f'' > 0 \) then \( f \) is concave upward, and if \( f'' < 0 \), then \( f \) is concave downward.
- A sign chart for \( f'' \) can be used to tell where \( f \) is concave upward or concave downward.
- An **inflection point** of \( f \) is a point \((c, f(c))\) on the graph of \( f \) where the concavity changes.

Point of Diminishing Returns

- If sales \( N(x) \) are expressed as a function of the amount \( x \) spent on advertising, then the dollar amount at which \( N'(x) \), the rate of change of sales, goes from increasing to decreasing is called the **point of diminishing returns**.
- If \( d \) is the point of diminishing returns, then \((d, N(d))\) is an inflection point of \( N(x) \).

4.3 L'Hôpital's Rule

0/0 Indeterminate Forms

- L'Hôpital's rule for \(0/0\) indeterminate forms states: if \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 \), then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

provided the second limit exists or is \(\infty\) or \(-\infty\).
- Always check to make sure that L'Hôpital's rule is applicable before using it.
- L'Hôpital's rule remains valid if the symbol \( x \to c \) is replaced everywhere it occurs by one of

\[
x \to c^+, \quad x \to c^-, \quad x \to \infty, \quad x \to -\infty
\]

\(\infty/\infty\) Indeterminate Forms

L'Hôpital's rule is also valid for indeterminate forms \(\pm\infty/\pm\infty\).
4.4 Curve-Sketching Techniques

**Graphing Strategy**

1. **Analyze** $f(x)$.
   - Find the domain of $f$.
   - Find the intercepts.
   - Find asymptotes.

2. **Analyze** $f'(x)$.
   - Find the partition numbers for $f'$ and the critical numbers of $f$.
   - Construct a sign chart for $f'(x)$.
   - Determine the intervals on which $f$ is increasing and decreasing.
   - Find the local maxima and minima of $f$.

3. **Analyze** $f''(x)$.
   - Find the partition numbers for $f''(x)$.
   - Construct a sign chart for $f''(x)$.
   - Determine the intervals on which the graph of $f$ is concave upward and concave downward.
   - Find the inflection points of $f$.

4. **Sketch the graph of** $f$.
   - Draw asymptotes and locate intercepts, local maxima and minima, and inflection points.
   - Sketch in what you know from steps 1-3.
   - Plot additional points as needed and complete the sketch.

**Oblique Asymptote**

If $f(x) = n(x)/d(x)$ is a rational function and the degree of $n(x)$ is 1 more than the degree of $d(x)$, then the graph of $f(x)$ has an **oblique asymptote** of the form $y = mx + b$. 
4.5 Absolute Maxima and Minima

**Absolute Extrema**
- If \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \), then \( f(c) \) is called the **absolute maximum** of \( f \).
- If \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \), then \( f(c) \) is called the **absolute minimum** of \( f \).
- An absolute maximum or absolute minimum is called an **absolute extremum**.
- Absolute extrema, if they exist, must occur at critical numbers or endpoints.
- To find the absolute maximum and absolute minimum of a continuous function \( f \) on a closed interval, identify the endpoints and critical numbers in the interval, evaluate the function \( f \) at each of them, and choose the largest and smallest values of \( f \).

**Second-Derivative Test for Local Extrema**
- If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f(c) \) is a local minimum.
- If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f(c) \) is a local maximum.
- No conclusion can be drawn if \( f''(c) = 0 \).

**Second-Derivative Test for Absolute Extrema on an Open Interval**
This is applicable when a function \( f \) is continuous on an open interval \( I \) and there is only one critical number \( c \) in \( I 
- If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f(c) \) is the absolute minimum of \( f \) on \( I \).
- If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f(c) \) is the absolute maximum of \( f \) on \( I \).

4.6 Optimization

**Strategy for Solving Optimization Problems**
1. Introduce variables, look for relationships among the variables, and construct a mathematical model of the form
   
   \( \text{Maximize (or minimize) } f(x) \text{ on the interval } I \).
2. Find the critical numbers of \( f(x) \).
3. Use the procedures developed in Section 4.5 to find the absolute maximum (or minimum) of \( f(x) \) on the interval \( I \) and the numbers \( x \) where this occurs.
4. Use the solution to the mathematical model to answer all the questions asked in the problem.
**End-Point Solution**

- If the absolute maximum or absolute minimum occurs at an endpoint, not a critical number in the interior of $I$, the extremum is called an **end-point solution**.