1 Functions and Graphs

1.1 Functions

Cartesian Coordinate System

- A **Cartesian or rectangular coordinate system** is formed by the intersection of a horizontal real number line, usually called the x **axis**, and a vertical real number line, usually called the y **axis**, at their origins
- The axes determine a plane and divide this plane into four **quadrants**.
- Each point in the plane corresponds to its **coordinates** - an ordered pair (a, b) determined by passing horizontal and vertical lines through the point
- The **abscissa** or x **coordinate** a is the coordinate of the intersection of the vertical line and the x axis.
- The **ordinate** or y **coordinate** b is the coordinate of the intersection of the horizontal line and the y axis.
- The point with coordinates (0, 0) is called the **origin**.

Plotting a Graph

- **Point-by-point plotting** may be used to **sketch the graph** of an equation in two variables.
- Plot enough points from its **solution set** in a rectangular coordinate system so that the total graph is apparent and then connect these points with a smooth curve.

Function

A **function** is a correspondence between two sets of elements such that to each element in the first set there corresponds one and only one element in the second set.

- The first set is called the **domain**.
- The set of corresponding elements in the second set is called the **range**

Dependent and Independent Variables

- If x is a placeholder for the elements in the domain of a function, then x is called the **independent variable** or the **input**.
- If y is a placeholder for the elements in the range, then y is called the **dependent variable** or the **output**.
- The symbol $f(x)$ represents the element in the range of f that corresponds to the element x of the domain.

Functions Specified by Equations

- If in an equation in two variables we get exactly one output for each input, then the equation specifies a function. The graph of such a function is just the graph of the specifying equation.
- If we get more than one output for a given input, then the equation does not specify a function.
- If the domain is not indicated, then the domain is the set of all inputs that produce real numbers as outputs.

Vertical-Line Test

- The **vertical-line test** can be used to determine whether or not an equation in two variables specifies a function.
- If each vertical line in the coordinate system passes through at most one point on the graph of an equation, then the equation specifies a function.
- If any vertical line passes through two or more points on the graph of an equation, then the equation does not specify a function.

Linear and Constant Functions

- Functions specified by equations of the form $y = mx + b$, where $m \neq 0$, are called **linear functions**.
- Functions specified by equations of the form $y = b$ are called **constant functions**.

Break-Even and Profit-Loss Analysis

Break-even and **profit-loss** analysis use a cost function C and a revenue function R to determine when a company will:

- have a loss $(R < C)$,
- break even $(R = C)$,
- have a profit $(R > C)$

1.2 Elementary Functions: Graphs and Transformations

Basic Elementary Functions

Six **basic elementary functions** are:

- the identity function
- the square and cube functions
- the square root and cube root functions
- the absolute value function.

Transformations

Performing an operation on a function produces a **transformation** of the graph of the function. The basic graph transformations are:

- **vertical and horizontal translations** (shifts)
- **reflection in the x axis**
- **vertical stretches and shrinks**.

Piecewise-Defined Functions

A **piecewise-defined function** is a function whose definition uses different rules for different parts of its domain.

1.3 Linear and Quadratic Functions

Mathematical Model

A **mathematical model** is a mathematics problem that, when solved, will provide information about a real-world problem.

Linear Equation in Two Variables

- A **linear equation in two variables** is an equation that can be written in the **standard form** $Ax + By = C$, where A, B and C are constants (A and B are not both zero), and x and y are variables.
- The graph of a linear equation in two variables is a line, and every line in a Cartesian coordinate system is the graph of an equation of the form $Ax + By = C$.

Slope

- If (x_1, y_1) and (x_2, y_2) are two points on a line with $x_1 \neq x_2$, then the **slope** of the line is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- The **point-slope form** of the line with slope m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- The **slope-intercept form** of the line with slope m that has y intercept b is $y = mx + b$.

Graphs of Vertical and Horizontal Lines

- The graph of the equation $x = a$ is a **vertical line**.
- The graph of the equation $y = b$ is a **horizontal line**.

LinearFunctions

- A function of the form $f(x) = mx + b$, where $m \neq 0$, is a **linear function**.

Quadratic Functions

- A function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is a **quadratic function** in **standard form**, and its graph is a **parabola**.
- Completing the square in the standard form of a quadratic function produces the **vertex form**

$$f(x) = a(x - h)^2 + k$$

- From the vertex form of a quadratic function, we can read off the vertex, axis of symmetry, maximum or minimum, and range, and can easily sketch its graph.

Equilibrium Point, Price, and Quantity

- In a competitive market, the intersection of the supply equation and the demand equation is called the **equilibrium point**.
- The corresponding price is called the **equilibrium price**.
- The common value of supply and demand is called the **equilibrium quantity**.

1.4 Polynomial and Rational Functions

Polynomial Functions

- A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a nonnegative integer called the **degree** of the polynomial.

- The coefficients $a_0, a_1, \ldots, a_n$ are real numbers with **leading coefficient** $a_n \neq 0$.
- The **domain** of a polynomial function is the set of all real numbers.
- The graph of a polynomial function of degree n can intersect the x axis at most n times. An x intercept is also called a **zero** or a **root**.
- The graph of a polynomial function has no sharp corners and is **continuous**, that is, it has no holes or breaks.

Rational Functions

- A **rational function** is any function that can be written in the form

$$f(x) = \frac{n(x)}{d(x)}$$

  where $n(x)$ and $d(x)$ are polynomials.

- The **domain** is the set of all real numbers such that $d(x) \neq 0$.
- Unlike polynomial functions, a rational function can have vertical asymptotes [but not more than the degree of the denominator $d(x)$] and at most one horizontal asymptote.

1.5 Exponential Functions

Exponential Functions

- An **exponential function** is a function of the form

$$f(x) = b^x$$

  where $b \neq 1$ is a positive constant called the **base**.

- The **domain** of f is the set of all real numbers.
- The **range** of f is the set of positive real numbers.

Graphs of Exponential Functions

The graph of an exponential function:

- is continuous
- passes through (0, 1)
- has the x axis as a horizontal asymptote
- increases as x increases if $b > 1$
- decreases as x increases if $0 < b < 1$

Properties of Exponential Functions

For a and b positive, $a \neq 1$, $b \neq 1$, and x and y real:

- $a^x a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$.
- $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.
- $a^x = a^y$ if and only if $x = y$.
- For $x \neq 0$, $a^x = b^x$ if and only if $a = b$.

The Number e

The base that is used most frequently in mathematics is the irrational number $e \approx 2.7183$.

Calculating Interest

Exponential functions are used in computations of **compound interest** and **continuous compound interest**:

- Compound interest: $A = P\left(1 + \dfrac{r}{m}\right)^{mt}$
- Continuous compound interest: $A = Pe^{rt}$

where:

- $A =$ amount (future value) at the end of t years
- $P =$ principal (present value)
- $r =$ annual rate (expressed as a decimal)
- $m =$ number of compounding periods per year
- $t =$ time in years

1.6 Logarithmic Functions

The Inverse of a Function

- A function is said to be **one-to-one** if each range value corresponds to exactly one domain value.
- The **inverse** of a one-to-one function f is the function formed by interchanging the independent and dependent variables of f. That is, (a, b) is a point on the graph of f if and only if (b, a) is a point on the graph of the inverse of f.
- A function that is not one-to-one does not have an inverse.

Logarithmic Functions

- The inverse of the exponential function with base b is called the **logarithmic function with base b**, denoted $y = \log_b x$.
- The **domain** of $\log_b x$ is the set of all positive real numbers (which is the range of b^x).
- The **range** of $\log_b x$ is the set of all real numbers (which is the domain of b^x).
- $y = \log_b x$ is equivalent to $x = b^y$.

Properties of Logarithmic Functions

If b, M and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

- $\log_b 1 = 0$ and $\log_b b = 1$.
- $\log_b b^x = x$ and $b^{\log_b x} = x, x > 0$.
- $\log_b MN = \log_b M + \log_b N$ and $\log_b \dfrac{M}{N} = \log_b M - \log_b N$.
- $\log_b M^p = p \log_b M$.
- $\log_b M = \log_b N$ if and only if $M = N$.

Common and Natural Logarithms

- Logarithms to the base 10 are called **common logarithms**, often denoted by $\log x$.
- Logarithms to the base e are called **natural logarithms**, often denoted by $\ln x$.

Doubling Time

Logarithms can be used to find an investment's **doubling time** - the length of time it takes for the value of an investment to double.

2 Limits and the Derivative

2.1 Introduction to Limits

The Graph of a Function

The graph of the function $y = f(x)$ is the graph of the set of all ordered pairs $(x, f(x))$.

Limit of a Function

- The limit of the function $y = f(x)$ as x approaches c is L, written as $\lim_{x \to c} f(x) = L$, if the functional value $f(x)$ is close to the single real number L whenever x is close to, but not equal, to c (on either side of c).
- The limit of the function $y = f(x)$ as x approaches c from the left is K, written as $\lim_{x \to c^-} f(x) = K$, if $f(x)$ is close to K whenever x is close to, but to the left of, c on the real number line.
- The limit of the function $y = f(x)$ as x approaches c from the right is L, written as $\lim_{x \to c^+} f(x) = L$, if $f(x)$ is close to L whenever x is close to, but to the right of, c on the real number line.

Limit of a Difference Quotient

- The limit of the difference quotient $[f(a + h) - f(a)]/h$ is often a 0/0 indeterminate form.
- Algebraic simplification is often required to evaluate this type of limit.

2.2 Infinite Limits and Limits at Infinity

Infinite Limits

If $f(x)$ increases or decreases without bound as x approaches a from either side of a, then the line $x = a$ is a **vertical asymptote** of the graph of $y = f(x)$.

Limits at Infinity

- If $f(x)$ gets close to L as x increases without bound or decreases without bound, then L is called the limit of f at ∞ or $-\infty$.
- The **end behavior** of a function is described by its limits at infinity.

Horizontal Asymptotes

- If $f(x)$ approaches L as $x \to \infty$ or as $x \to -\infty$, then the line $y = L$ is a **horizontal asymptote** of the graph of $y = f(x)$.
- Polynomial functions never have horizontal asymptotes.
- A rational function can have at most one horizontal asymptote.

2.3 Continuity

Continuous Functions

Intuitively, the graph of a continuous function can be drawn without lifting a pen off the paper. By definition, a function f is **continuous at c** if:

- $\lim_{x \to c} f(x)$ exists.
- $f(c)$ exists.
- $\lim_{x \to c} f(x) = f(c)$

Uses of Continuity Properties

- Continuity properties are useful for determining where a function is continuous and where it is discontinuous.
- Continuity properties are also useful for solving inequalities.

2.4 The Derivative

Average and Continuous Rates of Change

- Given a function $y = f(x)$, the **average rate of change** is the ratio of the change in y to the change in x.
- The **instantaneous rate of change** is the limit of the average rate of change as the change in x approaches 0.

Slope of a Graph

- The slope of the secant line through two points on the graph of a function $y = f(x)$ is the ratio of the change in y to the change in x.
- The **slope of the graph** at the point $(a, f(a))$ is the limit of the slope of the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$ as h approaches 0, provided the limit exists.
- In this case, the **tangent line** to the graph is the line through $(a, f(a))$ with slope equal to the limit.

The Derivative

- The **derivative of** $y = f(x)$ **at** x, denoted $f'(x)$, is the limit of the difference quotient $[f(x + h) - f(x)]/h$ as $h \to 0$ (if the limit exists).
- If the limit of the difference quotient does not exist at $x = a$, then f is **nondifferentiable at a** and $f'(a)$ does not exist.

The Four-Step Process to Find Derivatives

The four-step process for finding the derivative of a function f is:

- Find $f(x + h)$.
- Find $f(x + h) - f(x)$.
- Find $\dfrac{f(x + h) - f(x)}{h}$.
- Find $\displaystyle\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

2.5 Basic Differentiation Properties

Constant Function Rule

The derivative of a constant function is 0.

Power Rule

For any real number n, the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

Constant Multiple Property

If f is any differentiable function, then the derivative of $kf(x)$ is $kf'(x)$.

Sum and Difference Property

The derivative of the sum or difference of two differentiable functions is the sum or difference of the derivatives of the functions.

2.6 Differentials

Increment

- Given the function $y = f(x)$, the change in x is also called the **increment of** x and is denoted as Δx.
- The corresponding change in y is called the **increment of** y and is given by $\Delta y = f(x + \Delta x) - f(x)$.

Differential

- If $y = f(x)$ is differentiable at x, then the **differential of** x is $dx = \Delta x$.
- The **differential of** $y = f(x)$ is $dy = f'(x)dx$, or $df = f'(x)dx$. In this context, x and dx are both independent variables.

2.7 Marginal Analysis in Business and Economics

Marginal Cost

- If $y = C(x)$ is the total cost of producing x items, then $y = C'(x)$ is the **marginal cost** and $C(x+1) - C(x)$ is the exact cost of producing item $x+1$.
- Furthermore, $C'(x) \approx C(x+1) - C(x)$.
- Similar statements can be made regarding total revenue and total profit functions.

Average Cost

- If $y = C(x)$ is the total cost of producing x items, then the **average cost**, or cost per unit, is $\overline{C}(x) = \dfrac{C(x)}{x}$.
- The **marginal average cost** is $\overline{C}'(x) = \dfrac{d}{dx}\overline{C}(x)$.
- Similar statements can be made regarding total revenue and total profit functions.