MTH 131: Mathematical Analysis for Management, Fall 2017

Midterm 3 Answer Key

Name: ______________________________

Student Number: ______________________________

Answer the questions in the spaces provided on the question sheets. **Show all of your work.** If you run out of room for an answer, continue on the back of the page. You are allowed to use a non-graphing calculator. You are allowed a one page formula sheet.

Page	Points	Score
3	12	
4	15	
5	10	
6	20	
7	14	
8	9	
9	9	
10	11	
11	10	
Total:	110	

1. Find the following indefinite integral: $\int 10x^4\,dx$ [2]

Full solution:

$$\int 10x^4\,dx = 10\int x^4\,dx = 10\frac{x^5}{5} + C = 2x^5 + C$$

$\int 10x^4\,dx =$ $\underline{\quad 2x^5 + C \quad}$ (Use C as the arbitrary constant.)

2. Find the following indefinite integral: $\int 15e^u\,du$ [2]

Full solution:

$$\int 15e^u\,dx = 15\int e^u\,du = 15e^u + C$$

$\int 15e^u\,du =$ $\underline{\quad 15e^u + C \quad}$ (Use C as the arbitrary constant.)

3. Find the following indefinite integral: $\int 5x(3-x)\,dx$ [4]

Full solution:

$$\int 5x(3-x)\,dx = \int (15x - 5x^2)\,dx = \frac{15x^2}{2} - \frac{5x^3}{3} + C$$

$\int 5x(3-x)\,dx =$ $\underline{\quad 15x^2/2 - 5x^3/3 + C \quad}$ (Use C as the arbitrary constant.)

4. Find the equation of the curve that passes through (2, 3) if its slope is given by the following equation: [4]

$$\frac{dy}{dx} = 2x - 6$$

Full solution:

$$y = \int (2x-6)\,dx = x^2 - 6x + C$$
$$y(2) = 3 = 2^2 - 6(2) + C = -8 + C \Rightarrow C = 11$$
$$\Rightarrow x(t) = x^2 - 6x + 11$$

$y(x) =$ $\underline{\quad x^2 - 6x + 11 \quad}$

5. Find the indefinite integral: $\int (8x-3)^{-4}\, dx$ [5]

Full solution:

Use the method of substitution: $u = 8x - 3 \Rightarrow du = 8\, dx \Rightarrow\ dx = \dfrac{du}{8}$. Then:

$$\begin{aligned}\int (8x-3)^{-4}\, dx &= \int u^{-4}\frac{du}{8} = \frac{1}{8}\int u^{-4}\, du = \frac{1}{8}\frac{u^{-3}}{-3} + C = -\frac{u^{-3}}{24} + C = -\frac{1}{24u^3} + C \\ &= -\frac{1}{24(8x-3)^3} + C\end{aligned}$$

$\int (8x-3)^{-4}\, dx =$ $\underline{\quad -1/24(8x-3)^3 + C \quad}$ (Use C as the arbitrary constant.)

6. Find the indefinite integral: $\int x^{12} e^{x^{13}}\, dx$ [5]

Full solution:

Use the method of substitution: $u = x^{13} \Rightarrow du = 13x^{12}\, dx \Rightarrow x^{12}\, dx = \dfrac{du}{13}$. Then:

$$\int x^{12} e^{x^{13}}\, dx = \frac{1}{13}\int e^u\, du = \frac{1}{13}e^e + C = \frac{1}{13}e^{x^{13}} + C$$

$\int x^{12} e^{x^{13}}\, dx =$ $\underline{\quad e^{x^{13}}/13 + C \quad}$ (Use C as the arbitrary constant.)

7. Find the indefinite integral: $\int \frac{(\ln x)^6}{x}\, dx$ [5]

Full solution:

Use the method of substitution: $u = \ln x \Rightarrow du = 1/x\, dx$. Then:

$$\int \frac{(\ln x)^6}{x}\, dx = \int u^6\, du = \frac{u^7}{7} + C = \frac{(\ln x)^7}{7} + C$$

$\int \frac{(\ln x)^6}{x}\, dx =$ $\underline{\quad (\ln x)^7/7 + C \quad}$ (Use C as the arbitrary constant.)

8. Find the general solution for the first-order differential equation: $\frac{dy}{dx} = \frac{13}{x}$. [3]

Full solution:

$$\frac{dy}{dx} = \frac{13}{x} \Rightarrow y = \int \frac{13}{x}\, dx = 13 \ln|x| + C$$

$y =$ $13 \ln|x| + C$ (Use C as the arbitrary constant.)

9. Find the amount A in an account after t years given the following conditions: [5]

$$\frac{dA}{dt} = 0.02A; \qquad A(0) = 4000$$

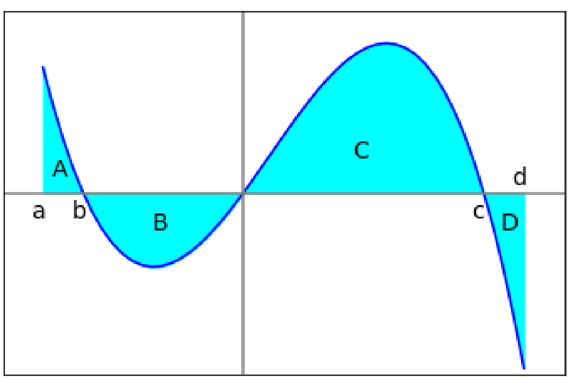
Full solution:

This is an exponential growth equation: $A_0 = 4000, r = 0.02 \Rightarrow A(t) = A_0 e^{rt} = 4000e^{0.02t}$.

$A(t) =$ $4000e^{0.02t}$

10. Calculate the definite integral below by referring to the figure with the indicated areas. [2]

$$\int_a^c f(x)\, dx$$



Area A = 1.213 Area B = 2.71

Area C = 5.369 Area D = 1.611

Full solution:

The definite integral is the area under the graph of the function in the interval $[a, c]$, which is the area $A - B + C = 3.872$.

$\int_a^c f(x)\, dx =$ **3.872** (Write the numerical value.)

11. Evaluate the integral: $\int_0^4 (3x^2+3)\,dx$. [5]

Full solution:

$$\int_0^4 (3x^2+3)\,dx = \left(x^3+3x\right)\Big|_0^4 = (4^3+3(4)) - (0^3+3(0)) = 76 - 0 = 76$$

$\int_0^4 (3x^2+3)\,dx =$ ______**76**______

12. Evaluate the integral: $\int_7^7 \left(x^2-7x+2\right)^{13}\,dx$ [2]

Full solution:

The limits of integration are the same, so the result is zero.

$\int_7^7 \left(x^2-7x+2\right)^{13}\,dx =$ ______**0**______

13. Calculate the definite integral: $\int_4^7 \frac{1}{x-1}\,dx$ [5]

Full solution:

$$\int_4^7 \frac{1}{x-1}\,dx = \ln|x-1|\Big|_4^7 = \ln 6 - \ln 3 = \ln 2$$

$\int_4^7 \frac{1}{x-1}\,dx =$ ______**ln 2 = 0.693**______

14. Find the area bounded by the curves $f(x) = x^2 - 3x$ and $g(x) = 2x + 6$. [8]

Full solution:

Set $f(x) = g(x)$ and solve for x:

$$x^2 - 3x = 2x + 6 \Rightarrow x^2 - 5x - 6 = 0 = (x+1)(x-6) \Rightarrow x = -1, 6$$

$f(x) \leq g(x)$ on [-1, 6], so the area is:

$$\begin{aligned} A &= \int_{-1}^6 [g(x) - f(x)]\,dx = \int_{-1}^6 (2x+6) - (x^2-3x)\,dx = \int_{-1}^6 5x + 6 - x^2\,dx \\ &= \left(\frac{5x^2}{2} + 6x - \frac{x^3}{3}\right)\Big|_{-1}^6 = \left(\frac{5(6^2)}{2} + 6(6) - \frac{6^3}{3}\right) - \left(\frac{5((-1)^2)}{2} + 6(-1) - \frac{(-1)^3}{3}\right) \\ &= 54 - (-3.167) = 57.167 \end{aligned}$$

The area, calculated to three decimal places, is ______**57.167**______ square units.

15. Find the future value at 6% interest, compounded continuously for 7 years, of the continuous income stream with rate of flow: [6]

$$f(t) = 800e^{-0.02t}$$

Full solution:

The future value is given by:

$$e^{rT}\int_0^T f(t)e^{-rt}\,dt = e^{0.06(7)}\int_0^7 800e^{-0.02t}e^{-0.06t}\,dt = e^{0.06(7)}\int_0^7 800e^{-0.08t}\,dt$$
$$= 800e^{0.42}\left(\frac{e^{-0.08t}}{-0.08}\right)\Bigg|_0^7 = 800e^{0.42}\left(\frac{e^{-0.56}-e^0}{-0.08}\right) = 6526.03$$

The future value is **\$6526** (Round to the nearest dollar.)

16. Find the consumers' surplus at a price level of $\bar{p} = \$110$ for the price-demand equation below: [6]

$$p = D(x) = 500 - 0.04x$$

Full solution:

$$\bar{p} = \$110 = 500 - 0.04\bar{x} \Rightarrow \bar{x} = \frac{500-110}{0.4} = 9750$$

The consumers' surplus is then:

$$CS = \int_0^{\bar{x}}[D(x)-\bar{p}]\,dx = \int_0^{9750}[500-0.04x-110]\,dx = \int_0^{9750}[390-0.04x]\,dx$$
$$= \left(390x - 0.02x^2\right)\Bigg|_0^{9750} = (390(9750) - 0.02(9750^2) = 1,901,250$$

The consumers' surplus is **\$1,901,250** (Round to the nearest dollar.)

17. Find the value $g(-1,1)$ of the function $g(x,y) = \dfrac{20}{x^2+3y}$. [2]

Full solution:

$$g(-1,1) = \frac{20}{(-1)^2+3(1)} = \frac{20}{4} = 5$$

√ $g(-1,1) =$ **5**

◯ The function $g(-1,1)$ is undefined.

18. The productivity of a certain country with the utilization of x units of labor and y units of capital is given approximately by the function $f(x,y) = 100x^{0.74}y^{0.26}$. Find $f_x(x,y)$ and $f_y(x,y)$

Full solution:

$$f_x(x,y) = 100(0.74)x^{0.74-1}y^{0.26} = 74x^{-0.26}y^{0.26}$$

(a) $f_x(x,y) =$ $74x^{-0.26}y^{0.26}$ [2]

Full solution:

$$f_y(x,y) = 100x^{0.74}(0.26)y^{0.26-1} = 26x^{0.74}y^{-0.74}$$

(b) $f_y(x,y) =$ $26x^{0.74}y^{-0.74}$ [2]

19. Find $f_x(x,y)$ and $f_y(x,y)$, and explain why $f(x,y)$ has no local extrema:

$$f(x,y) = 5x + 7y + 8$$

Full solution:

Full solution:

$$f_x(x,y) = 5 + 0 + 0 = 5$$

(a) $f_x(x,y) =$ **5** [2]

Full solution:

$$f_y(x,y) = 0 + 7 + 0 = 7$$

(b) $f_y(x,y) =$ **7** [2]

(c) Choose the best explanation below for the reason that $f(x,y)$ has no local extrema. [1]

Full solution:

The function has no critical points, since $f_x(x,y)$ and $f_y(x,y)$ are non-zero for all (x,y).

- ◯ The functions $f_x(x,y)$ and $f_y(x,y)$ have no local extrema.
- ◯ The second derivative test fails because $AC - B^2 = 0$.
- ◯ The functions $f_x(x,y)$ and $f_y(x,y)$ are never equal to each other.
- √ **The functions $f_x(x,y)$ and $f_y(x,y)$ are non-zero for all (x,y).**

20. Find the critical points of the following function, then use the second derivative test to determine if the critical points are local extrema.

$$f(x, y) = -3x^2 + 2xy - 2y^2 + 14x + 2y + 10$$

(a) Find the critical points of $f(x, y)$. Use a comma to separate answers as needed. [6]

Full solution:

The critical points are where $f_x(x, y) = 0$ and $f_y(x, y) = 0$:

$$f_x(x, y) = -6x + 2y + 14 = 0 \qquad \text{and} \qquad f_y(x, y) = 2x - 4y + 2 = 0$$

These are two linear equations in two variables, so we need to find where they intersect. Add $2f_x$ to f_y to eliminate y:

$$\begin{aligned} 2f_x + f_y = 0 &= 2(-6x + 2y + 14) + (2x - 4y + 2) \\ &= -12x + 4y + 28 + 2x - 4y + 2 \\ &= -10x + 30 \Rightarrow x = 3 \\ f_x(x, y) &= -6x + 2y + 14 = 0 \Rightarrow y = 2 \end{aligned}$$

So, the function has a single critical point at (3, 2)

The function $f(x, y)$ has critical points at **(3, 2)**

Use the second derivative test to determine if the critical points are local extrema.

Full solution:

Applying the second-derivative test:

$$\begin{aligned} A &= f_{xx}(x, y) = -6 \\ B &= f_{xy}(x, y) = 2 \\ C &= f_{yy}(x, y) = -4 \\ AC - B^2 &= (-6)(-4) - 2^2 = 20 > 0 \text{ for all } (x, y) \end{aligned}$$

Since $A = -6 < 0$, any critical points are local maxima by the second derivative test.

(b) Find the location of local minima. [1]

○ The function has local minima at __________

√ **The function has no local minima.**

(c) Find the location of local maxima. [1]

√ **The function has local maxima at (3, 2)**

○ The function has no local maxima.

(d) Find the location of any saddle points. [1]

○ The function has saddle points at __________

$\surd$ **The function has no saddle points.**

(e) Find the location of any critical points where the second-derivative test fails. [1]

◯ The second-derivative test fails at ______________________

$\surd$ **The second derivative test does not fail for any of the critical points.**

21. Use the method of Lagrange Multipliers to minimize $f(x, y) = x^2+y^2$ subject to $2x+4y = 30$ [10]

Full solution:

The constraint is $g(x, y) = 2x + 4y - 30 = 0$. Form the function:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x^2 + y^2 + \lambda(2x + 4y - 30)$$

The critical points of this function are where:

$$\begin{aligned} F_x &= 2x + 2\lambda = 0 \Rightarrow \lambda = -x \\ F_y &= 2y + 4\lambda = 0 \Rightarrow \lambda = -y/2 \\ F_\lambda &= 2x + 4y - 30 = 0 \end{aligned}$$

So, $-x = -y/2$, therefore $y = 2x$. Substituting into $F_\lambda = 0$ yields $2x + 4(2x) - 30 = 10x - 30 = 0 \Rightarrow x = 3, y = 6$. The maximum value is then $f(3, 6) = 3^2 + 6^2 = 45$.

$x =$ ______**3**______

Full solution:

$y =$ ______**6**______

Full solution:

The value of f at the minimum is ______**45**______

EXTRA CREDIT QUESTION

22. Find the Gini index of income concentration for the Lorenz curve with equation: [10]

$$f(x) = xe^{x-1}$$

Full solution:

The Gini index is:

$$GI = 2\int_0^1 [x - f(x)]\,dx = 2\int_0^1 [x - xe^{x-1}]\,dx = 2\int_0^1 x\,dx - 2\int_0^1 xe^{x-1}\,dx$$

Integrate by parts to find an antiderivative for the second integral:

$$u = x \Rightarrow du = dx \qquad \text{and} \qquad dv = e^{x-1}\,dx \Rightarrow v = e^{x-1}$$

Use the integration by parts formula:

$$\int u\,dv = uv - \int v\,du = xe^{x-1} - \int e^{x-1}\,dx = xe^{x-1} - e^{x-1}$$

Now evaluate the definite integral using the antiderivative above:

$$\begin{aligned}
2\int_0^1 x\,dx - 2\int_0^1 xe^{x-1}\,dx &= 2\frac{x^2}{2}\bigg|_0^1 - 2\left(xe^{x-1} - e^{x-1}\right)\bigg|_0^1 \\
&= 2\left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 2\left[\left(e^0 - e^0\right) - \left(0 - e^{-1}\right)\right] \\
&= 2(1/2) - 2(e^{-1}) \\
&= 1 - \frac{2}{e} \\
&\approx 0.264
\end{aligned}$$

The Gini index is **0.264**