MTH 131: Mathematical Analysis for Management, Fall 2017

Midterm 2

Name: ______________________________

Student Number: ______________________________

Answer the questions in the spaces provided on the question sheets.

Show all of your work.

If you run out of room for an answer, continue on the back of the page.

You are allowed to use a non-graphing calculator.

You are allowed a one page formula sheet.

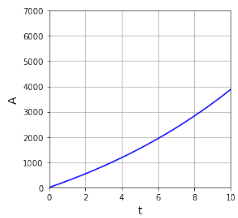
Page	Points	Score
3	10	
4	9	
5	7	
6	9	
7	8	
8	9	
9	5	
10	11	
11	12	
12	6	
13	6	
14	8	
Total:	100	

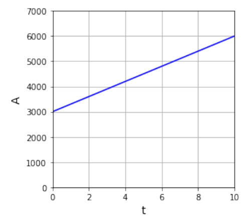
1. If \$3000 is invested at 8.3% compounded continuously, graph the amount in the account as a function of time for a period of 10 years. [2]

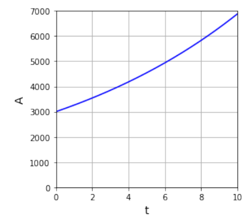
   Choose the correct graph.

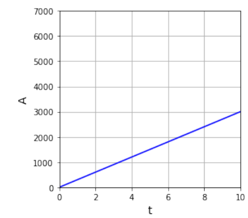
   Full solution:

   Choose the graph with a y-intercept of \$3000 that shows exponential growth.

   ○ 

   ○ 

   √ 

   ○ 

2. Recently, a certain bank offered a 5-year CD that earns 6.51% compounded continuously.

   (a) If \$10,000 is invested in this CD, how much will it be worth in 5 years? [4]

   Full solution:

$$\begin{aligned} A &= Pe^{rt} & \qquad P = 10,000, r = 0.0651, t = 5 \\ &= 10000e^{0.0651(5)} \\ &= 83112.69 \end{aligned}$$

   Approximately \$ **13847.23** (Round to the nearest cent)

   (b) How long will it take for the account to be worth \$25,000? [4]

   Full solution:

$$\begin{aligned} A &= Pe^{rt} & \qquad A = 25,000, P = 10,000, r = 0.0651 \\ 25000 &= 10000e^{0.0651t} \\ \frac{25000}{10000} &= e^{0.0651t} \\ 2.5 &= e^{0.0651t} \\ \ln(2.5) &= 0.0651t \\ t &= \ln(2.5)/0.0651 \\ &= 14.08 \end{aligned}$$

   Approximately **14.08** years (Round to two decimal places as needed).

3. Find $f'(x)$ for $f(x) = 7e^x - \dfrac{1}{x^8} + 5\ln x$. [3]

Full solution:
$f'(x) = \dfrac{d}{dx}(7e^x - \dfrac{1}{x^8} + 5\ln x) = 7\dfrac{d}{dx}e^x - \dfrac{d}{dx}x^{-8} + 5\dfrac{d}{dx}\ln x = 7e^x + \dfrac{8}{x^9} + \dfrac{5}{x}$

$f'(x) =$ $\underline{\quad 7e^x + 8/x^9 + 5/x \quad}$

4. Find $\dfrac{dy}{dx}$ for $y = 4\log_3 x$. [2]

Full solution:
$\dfrac{d}{dx}\log_b x = \dfrac{1}{x\ln b}$, so $\dfrac{d}{dx}4\log_3 x = 4\dfrac{1}{x\ln 3} = \dfrac{4}{x\ln 3}$

$\dfrac{dy}{dx} =$ $\underline{\quad 4/(x\ln 3) \quad}$

5. Find $\dfrac{dy}{dx}$ for $y = 23^x$. [2]

Full solution:
$\dfrac{d}{dx}b^x = b^x\ln b$, so $\dfrac{d}{dx}23^x = 23^x\ln 23$

$\dfrac{dy}{dx} =$ $\underline{\quad 23^x\ln 23 \quad}$

6. Find $f'(x)$ for $f(x) = 12x^2e^x$. [2]

Full solution:

By product rule,

$$\frac{d}{dx}(12x^2)(e^x) = 12x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(12x^2) = 12x^2(e^x) + e^x(24x) = 12xe^x(x+2)$$

$f'(x) =$ $\underline{\quad 12x^2e^x + 24xe^x \quad}$

7. Use the quotient rule to find the derivative of $y = \dfrac{6x^2+5}{x^2+4}$ [3]

Full solution:

By quotient rule,

$$\begin{aligned}\frac{d}{dx}\frac{6x^2+5}{x^2+4} &= \frac{(x^2+4)\frac{d}{dx}(6x^2+5)-(6x^2+5)\frac{d}{dx}(x^2+4)}{(x^2+4)^2}\\ &= \frac{(x^2+4)(12x)-(6x^2+5)(2x)}{(x^2+4)^2}\\ &= \frac{(12x^3+48x)-(18x^3+10x)}{(x^2+4)^2}\\ &= \frac{38x}{(x^2+4)^2}\end{aligned}$$

$y' =$ $\underline{\quad 38x/(x^2+4)^2 \quad}$

8. Find $f'(x)$ for $f(x) = (5-8\sqrt{x})^{10}$. [2]

Full solution:

By general power rule, $\dfrac{d}{dx}[u(x)]^n = n[u(x)]^{n-1}u'(x)$. Let $u(x) = 5-8\sqrt{x}$, so $u'(x) = \dfrac{d}{dx}(5-8x^{1/2} = -4x^{-1/2})$, and:

$$\frac{d}{dx}(5-8\sqrt{x})^{10} = 10(5-8\sqrt{x})^9(-4x^{-1/2}) = -40(5-8\sqrt{x})^9/\sqrt{x}$$

$f'(x) =$ $\underline{\quad -40(5-8\sqrt{x})^9/\sqrt{x} \quad}$

9. Find $f'(x)$ for $f(x) = 7\ln(5+6x^2)$. [2]

Full solution:

By chain rule, $\dfrac{d}{dx}\ln[f(x)] = \dfrac{1}{f(x)}f'(x)$. Let $f(x) = 5+6x^2$, so:

$$\frac{d}{dx}7\ln(5+6x^2) = 7\frac{1}{5+6x^2}\frac{d}{dx}(5+6x^2) = \frac{7}{5+6x^2}(12x) = \frac{84x}{5+6x^2}$$

$f'(x) =$ $\underline{\quad 84x/(5+6x^2) \quad}$

10. For $f(x) = \frac{1}{6}e^{2x^3-9x^2+12x+1}$:

(a) Find $f'(x)$. [2]

Full solution:

By chain rule, $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$. Let $f(x) = 2x^3 - 9x^2 + 12x + 1$, so

$$\begin{aligned}\frac{d}{dx}\frac{1}{6}e^{2x^3-9x^2+12x+1} &= \frac{1}{6}e^{2x^3-9x^2+12x+1}\frac{d}{dx}(2x^3 - 9x^2 + 12x + 1)\\ &= \frac{1}{6}e^{2x^3-9x^2+12x+1}(6x^2 - 18x + 12)\\ &= (x^2 - 3x + 2)e^{2x^3-9x^2+12x+1}\\ &= (x-1)(x-2)e^{2x^3-9x^2+12x+1}\end{aligned}$$

$f'(x) =$ $\underline{(x^2 - 3x + 2)e^{2x^3-9x^2+12x+1}}$

(b) Find the equation of the tangent line to the graph of f when $x = 0$ [3]

Full solution:

$$\begin{aligned}f(0) &= \frac{1}{6}e^{2(0)^3-9(0)^2+12(0)+1} &&= e/6\\ f'(0) &= ((0)^2 - 3(0) + 2)e^{2(0)^3-9(0)^2+12(0)+1} &&= 2e\end{aligned}$$

Plugging into the slope-intercept form $y = mx + b$ with $m = 2e$, $b = e/6$, yields

$$y = 2ex + e/6$$

$y =$ $\underline{y = 2ex + e/6}$

(c) Find the value(s) of x where the tangent line is horizontal. [2]

Full solution:

The tangent line is horizontal when $f'(x) = 0$, so $(x-1)(x-2)e^{2x^3-9x^2+12x+1} = 0$, Since $e^{2x^3-9x^2+12x+1} > 0$ for all x, this means $(x-1)(x-2) = 0$, so $x = 1$ or $x = 2$.

√ **The tangent line is horizontal at x =** **1, 2**

◯ The tangent line is never horizontal.

11. If it is possible to solve for y in terms of x, do so: $4x + 6y = e^y$. [2]

Full solution:

Can't be done.

◯ y = ________________

√ **It is impossible to solve the equation for y in terms of x.**

12. For the equation $y^2 + 4y + 5x = 0$:

(a) Use implicit differentiation to find y'. [4]

Full solution:

Differentiate each term with respect to x, using the chain rule for y^2:

$$\frac{d}{dx}(y^2 + 4y + 5x) = \frac{d}{dy}(y^2)\frac{dy}{dx} + 4\frac{dy}{dx} + 5\frac{d}{dx}x = 2yy' + 4y' + 5 = 0$$

Solve for y':

$$2yy' + 4y' + 5 = 0 \Rightarrow (2y+4)y' = -5 \Rightarrow y' = \frac{-5}{2y+4}$$

$y' =$ $\underline{\quad -5/(2y+4) \quad}$

(b) Evaluate y' at the point (-1, 1). [1]

Full solution:

$$\frac{-5}{2y+4} = \frac{-5}{2(1)+4} = -\frac{5}{6}$$

$y'(-1, 1) =$ $\underline{\quad -5/6 \quad}$

13. Assume that $x = x(t)$ and $y = y(t)$. Find $\frac{dx}{dt}$ using the following information: [3]

$x^2 + y^2 = 5.8$; $\frac{dy}{dt} = -2$ when $x = -1.8$ and $y = 1.6$.

Full solution:

Differentiate each term with respect to t, using the chain rule for x^2 and y^2:

$$\frac{d}{dx}(x^2)\frac{dx}{dt} + \frac{d}{dy}(y^2)\frac{dy}{dt} = 0 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Find dx/dt when $dy/dt = -2$, $x = -1.8$, and $y = 1.6$:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2(-1.8)\frac{dx}{dt} + 2(1.6)(-2) = 0 \Rightarrow \frac{dx}{dt} = -3.2/1.8 = -\frac{16}{9} = 1.\bar{7}$$

$\frac{dx}{dt} =$ $\underline{\quad -16/9 = 1.\bar{7} \quad}$

14. For $f(x) = 195 + 54x$:

(a) Find the percentage rate of change of $f(x)$. [3]

Full solution:

The percentage rate of change of $f(x)$ is

$$100f'(x)/f(x) = 100\frac{54}{195+54x} = \frac{5400}{195+54x}$$

The percentage rate of change is $\underline{\quad 5400/(195+54x) \quad}$

(b) Evaluate the percentage rate of change of $f(x)$ when $x = 5$. [1]

Full solution:

$$\frac{5400}{195+54x} = \frac{5400}{195+54(5)} = \frac{5400}{465} = 11.6\%$$

The percentage rate of change when $x = 5$ is <u>**11.6**</u> %. (Round to 1 decimal place)

15. Using the price-demand equation $x = f(p) = 15,000 - 650p$:

(a) Find $E(p)$, the elasticity of demand. [3]

Full solution:

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-650)}{15000-650p} = \frac{13p}{300-13p}$$

$E(p) = \underline{\quad 13p/(300-13p) \quad}$

(b) Evaluate $E(p)$ when $p = 20$. [1]

Full solution:

$$E(20) = \frac{13(20)}{300-13(20)} = \frac{260}{40} = 6.5$$

$E(20) =$ <u>**6.5**</u>

(c) Is the demand inelastic, elastic, or have unit elasticity when $p = 20$? [1]

Full solution:

Demand is inelastic if $0 < E(p) < 1$, elastic if $E(p) > 1$, and unit if $E(p) = 1$.

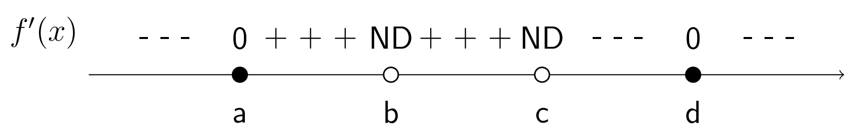
○ Inelastic
√ **Elastic**
○ Unit elasticity

(d) If prices are increased when $p = 20$, will revenues increase, decrease, or stay the same? [1]

Full solution:
When demand is elastic, price increases cause revenues to decrease.

○ Revenues will increase
√ **Revenues will decrease**
○ Revenues will stay the same

16. $f(x)$ is continuous on $(-\infty, \infty)$ and has critical numbers at $x = a$, b, c, and d. Use the sign chart below for $f'(x)$ to determine whether f has a local maximum, a local minimum, or neither at each critical number.



(a) Does $f(x)$ have a local minimum, a local maximum, or no local extremum at $x = a$? [1]

Full solution:
$f(x)$ changes from decreasing to increasing at $x = c$, so this is a local minimum.

√ **a local minimum** ○ a local maximum ○ no local extremum

(b) Does $f(x)$ have a local minimum, a local maximum, or no local extremum at $x = b$? [1]

Full solution:
$f'(x)$ does not change sign at $x = a$, so this is not a local extremum.

○ a local minimum ○ a local maximum √ **no local extremum**

(c) Does $f(x)$ have a local minimum, a local maximum, or no local extremum at $x = c$? [1]

Full solution:
$f(x)$ changes from increasing to decreasing at $x = b$, so this is a local maximum.

○ a local minimum √ **a local maximum** ○ no local extremum

(d) Does $f(x)$ have a local minimum, a local maximum, or no local extremum at $x = d$? [1]

Full solution:
$f'(x)$ does not change sign at $x = a$, so this is not a local extremum.

○ a local minimum ○ a local maximum √ **no local extremum**

17. Find the intervals on which $f(x)$ is increasing, the intervals on which $f(x)$ is decreasing, and the local extrema for $f(x) = -3x^2 - 24x - 20$.

Type your answers using interval notation, and use a comma to separate answers as needed.

(a) Where is $f(x)$ increasing? [3]

Full solution:
The function is increasing when $f'(x) > 0$. Since $f'(x) = -6x - 24 = -6(x + 4)$, then:

$$f'(x) > 0 \Rightarrow -6(x+4) > 0 \Rightarrow x + 4 < 0 \Rightarrow x < -4$$

√ **The function is increasing on** ______**($-\infty$, -4)**______

◯ There is no solution.

(b) Where is $f(x)$ decreasing? [3]

Full solution:
The function is decreasing when $f'(x) < 0$:

$$f'(x) < 0 \Rightarrow -6(x+4) < 0 \Rightarrow x + 4 > 0 \Rightarrow x > -4$$

√ **The function is decreasing on** ______**(-4, ∞)**______

◯ There is no solution.

(c) Which statement is true regarding the local extrema? [2]

Full solution:
The function changes increasing $\rightarrow$ decreasing at $x = -4$, so this is a local maximum.

◯ The function has a local minimum at $x =$ ______________

√ **The function has a local maximum at** $x =$ ______**-4**______

◯ The function has no local extrema.

18. Find the second derivative for the function $f(x) = 2x^3 + 3\ln x + e^x$. [3]

Full solution:

$$\begin{aligned} f(x) &= 2x^3 + 3\ln x + e^x \\ f'(x) &= 6x^2 + \frac{3}{x} + e^x &&= 6x^2 + 3x^{-1} + e^x \\ f''(x) &= 12x - \frac{3}{x^2} + e^x &&= 12x - 3x^{-2} + e^x \end{aligned}$$

$f''(x) =$ ______$12x - 3/x^2 + e^x$______

19. For the function $f(x) = x^3 - 3x^2 + 3x - 1$, find the intervals on which the graph of f is concave upward, the intervals on which the graph of f is concave downward, and the inflection points.

(a) For what interval(s) of x is the graph of f concave upward? [3]

Full solution:

Concavity depends on the sign of the second derivative:

$$f(x) = x^3 - 3x^2 + 3x - 1 \quad \Rightarrow f'(x) = 3x^2 - 6x + 3 \quad \Rightarrow f''(x) = 6x - 6 = 6(x-1)$$

$f''(x) > 0$ when $x > 1$, so the graph of f is concave upward for $x > 1$.

√ **The graph is concave upward on the interval(s)** **$(1, \infty)$**

◯ The graph is never concave upward.

(b) For what interval(s) of x is the graph of f concave downward? [3]

Full solution:

$f''(x) < 0$ for $x < 1$, so the graph of f is concave downward for $x < 1$.

√ **The graph is concave downward on the interval(s)** **$(-\infty, 1)$**

◯ The graph is never concave downward.

(c) Determine the x-coordinates of any inflection points of the graph of $f(x)$. Use a comma to separate your answers. [2]

Full solution:

Concavity changes when $x = 1$, so there is an inflection point when $x = 1$.

√ **There are inflection points at $x =$** **1**

◯ There are no inflection points.

20. Use L'Hôpital's rule to find the limit $\lim_{x \to 9} \dfrac{x^2 - x - 72}{x - 9}$. Use $-\infty$ and ∞ when appropriate. [4]

Full solution:

$$\lim_{x \to 9} x^2 - x - 72 = 9^2 - 9 - 72 = 0 \quad \text{and} \quad \lim_{x \to 0} x - 9 = 9 - 9 = 0$$

So, L'Hôpital's rule applies:

$$\lim_{x \to 0} \lim_{x \to 9} \frac{x^2 - x - 72}{x - 9} \overset{\text{L'Hôp}}{=} \lim_{x \to 9} \frac{\frac{d}{dx}(x^2 - x - 72)}{\frac{d}{dx}(x - 9)} = \lim_{x \to 9} \frac{2x - 1}{1} = \frac{2(9) - 1}{1} = 17$$

√ $\lim_{x \to 9} \dfrac{x^2 - x - 72}{x - 9} =$ **17**

◯ The limit does not exist.

21. Use L'Hôpital's rule to find the limit $\lim_{x\to 0} \dfrac{e^{3x}-1}{2x}$. Use $-\infty$ and ∞ when appropriate. [4]

Full solution:

$$\lim_{x\to 0} e^{3x} - 1 = e^{3(0)} - 1 = 0 \quad \text{and} \quad \lim_{x\to 0} 2x = 2(0) = 0$$

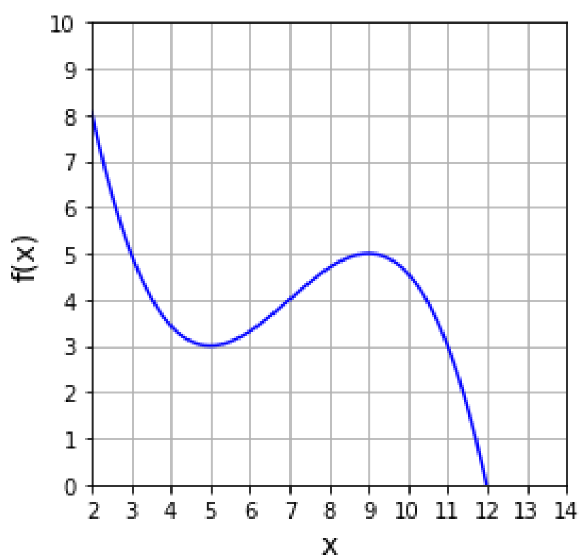
So, L'Hôpital's rule applies:

$$\lim_{x\to 0} \frac{e^{3x}-1}{2x} \overset{\text{L'Hôp}}{=} \lim_{x\to 0} \frac{\frac{d}{dx}(e^{3x}-1)}{\frac{d}{dx}(2x)} = \lim_{x\to 0} \frac{3e^{3x}}{2} = \frac{3e^0}{2} = \frac{3}{2}$$

√ $\lim_{x\to 0} \dfrac{e^{3x}-1}{2x} =$ **3/2**

◯ The limit does not exist.

22. Refer to the graph shown below. Find the absolute minimum and the absolute maximum over the interval [7, 12]. Round to the nearest integer.



(a) Identify the absolute minimum. [2]

Full solution:
From the graph, the smallest value of $f(x)$ over [7, 12] is when $f(x) = 0$ and $x = 12$.

√ **The absolute minimum is** **0** **at** $x =$ **12**

◯ There is no absolute minimum.

(b) Identify the absolute maximum. [2]

Full solution:
From the graph, the largest value of $f(x)$ over [7, 12] is when $f(x) = 5$ and $x = 9$.

$\surd$ **The absolute maximum is _____5_____ at** $x =$ **_____9_____**

◯ There is no absolute maximum.

23. Find the absolute maximum and absolute minimum values of the function $f(x) = x^2 - 10x - 8$ over the interval [1, 8], and indicate the x-values at which they occur.

(a) Identify the absolute maximum. [2]

Full solution:
The absolute maximum and absolute minimum values will occur either at critical points of f or at endpoints of the closed interval [0, 7]. We have:

$$f(x) = x^2 - 10x - 8 \quad \Rightarrow f'(x) = 2x - 10 = 2(x - 5)$$

The only critical point is at $x = 5$. Check the value the function at this point and the endpoints:

x	1	5	8
f(x)	-17	-33	-24

From the table, the maximum value of f is when $x = 1$.

The absolute maximum value is _____**-17**_____ at $x =$ _____**1**_____

(b) Identify the absolute minimum. [2]

Full solution:
From the table, the minimum value of f is when $x = 5$.

The absolute minimum value is _____**-33**_____ at $x =$ _____**5**_____

24. A wall is to be built to enclose a rectangular area of 300 square feet. The wall along three sides is to be built of brick that costs \$4 per foot. The fourth wall is to be built of wood that costs \$2 per foot. Find the dimensions of the rectangle that will allow for the least expensive wall to be built, and the total cost of the wall.

Full solution:

1. Let the side made of wood costing \$2 per foot be x, and the other side be y. Then:

$$\text{Area } = xy = 300 \quad \Rightarrow y = \frac{300}{x} \quad x > 0, \quad y > 0$$

   and we want to minimize the total cost:

$$\text{Cost } = 2x + 4x + 4y + 4y = 6x + 8y = 6x + 8\frac{300}{x} = 6x + \frac{2400}{x} = C(x)$$

2. Find the critical points of $C(x)$:

$$C'(x) = 6 - \frac{2400}{x^2} \quad \text{so} \quad C'(x) = 0 \Rightarrow x^2 = \frac{2400}{6} = 400 \Rightarrow x = 20$$

3. By the second-derivative test:

$$C''(x) = \frac{d}{dx}\left(6 - 2400x^{-2}\right) = -2400(-2)x^{-3} = \frac{24800}{x^3}$$

   Since $C''(x) > 0$ for all $x > 0$, the function $C(x)$ has an absolute minimum when $x = 20$ and $y = 300/20 = 15$.

4. The wood side is therefore 20 ft and the other side is 15 ft.

5. The total cost of the wall is $C(20) = 6(20) + \dfrac{2400}{20} = 120 + 120 = \$240$.

(a) The wall made of wood is ____**20**____ ft long and the other side is ____**15**____ ft long. [6]

(b) The total cost of the wall is \$ ____**240**____ [2]