Point Estimator

A *point estimator* is any function $W(\mathbf{X}) = W(X_1, \dots, X_n)$ of a sample $\mathbf{X}$; that is, any statistic is a point estimator.

Bias of a Point Estimator

The *bias* of a point estimator W of a parameter θ is $\text{Bias}_\theta W = \text{E}_\theta W - \theta$. An estimator whose bias is identically zero is called *unbiased* and satisfies $\text{E}_\theta W = \theta$ for all θ.

Likelihood Function

Given that $\mathbf{X} = \mathbf{x}$ is observed, the *likelihood function* is the function of θ defined by

$$L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$$

Mean Squared Error (MSE)

The *mean squared error* of an estimator W of a parameter θ is the function of θ

$$\text{MSE} = \text{E}_\theta(W-\theta)^2 = \text{Var}_\theta W + (\text{Bias}_\theta W)^2$$

Best Unbiased Estimator(UMVUE)

A *best* or *uniform minimum variance unbiased estimator* W^* of $\tau(\theta)$ satisfies $\text{E}_\theta W^* = \tau(\theta)$ and $\text{Var}_\theta W^* \leq \text{Var}_\theta W$ for any other unbiased estimator W, for all θ.

Maximum Likelihood Estimator

A *maximum likelihood estimator* (MLE) of the parameter θ based on a sample $\mathbf{X}$ is a parameter value $\hat{\theta}(\mathbf{X})$ at which $L(\theta|\mathbf{x})$ attains its maximum as a function of θ.

Cramér-Rao Inequality

If $\frac{d}{d\theta}\text{E}_\theta W(\mathbf{X}) = \int_{\mathcal{X}} \frac{\partial}{\partial\theta}[W(\mathbf{X}) f(\mathbf{x}|\theta)]\, d\mathbf{x}$ and $\text{Var}_\theta W(\mathbf{X}) < \infty$ then

$$\text{Var}_\theta(W(\mathbf{X})) \geq \frac{\left(\frac{d}{d\theta}\text{E}_\theta W(\mathbf{X})\right)^2}{\text{E}_\theta\left(\left(\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta)\right)^2\right)}$$

Best Unbiased Estimators are Unique

If W is a best unbiased estimator of $\tau(\theta)$, then W is unique.

Invariance Property of MLEs

If $\hat{\theta}$ is the MLE of θ, then for any function $\tau(\theta)$, the MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$.

Cramér-Rao Inequality, iid Case

If $X_1, \dots, X_n$ are iid with pdf $f(x|\theta)$, then

$$\text{Var}_\theta(W(\mathbf{X})) \geq \frac{\left(\frac{d}{d\theta}\text{E}_\theta W(\mathbf{X})\right)^2}{n\text{E}_\theta\left(\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right)}$$

Unbiased Estimators of 0

If $\text{E}_\theta W = \tau(\theta)$, W is the best unbiased estimator of $\tau(\theta)$ if and only if W is uncorrelated with all unbiased estimators of 0.

Best Estimator of Expected Value

Let T be a complete sufficient statistic for a parameter θ, and let $\phi(T)$ be any estimator based only on T. Then,

$\phi(T)$ is the unique best unbiased estimator of its expected value.

Attains Cramér-Rao Lower Bound

If $W(\mathbf{X})$ is any unbiased estimator of $\tau(\theta)$, then $W(\mathbf{X})$ attains the CRLB if and only if

$$a(\theta)[W(\mathbf{x}) - \tau(\theta)] = \frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x})$$

for some function $a(\theta)$.

Rao-Blackwell Theorem

Let $\text{E}_\theta W = \tau(\theta)$ for all θ and T be sufficient for θ. Define $\phi(T) = \text{E}(W|T)$. Then

$$\phi(T) = \tau(\theta) \text{ and } \text{Var}_\theta \phi(T) \leq \text{Var}_\theta W$$

for all θ; that is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

Lehmann-Scheffé Theorem

Unbiased estimators based on complete sufficient statistics are unique.