The Multivariate Normal Distribution

Let $\mathbf{Y}$ be a $n \times 1$ random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. We say that $\mathbf{Y}$ follows a **multivariate normal distribution** $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if:

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{-\tfrac{1}{2}(\mathbf{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\}$$

Let $\mathbf{A}$ be $m \times 1$ and $\mathbf{B}$ be $m \times n$ constant. Then:

$$\mathbf{A} + \mathbf{B}\mathbf{Y} \sim N_m(\mathbf{A} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}') \qquad (11.2.2)$$

Vector Differentiation/Least Squares

Let $\mathbf{a}$ and $\mathbf{x}$ be vectors and $\mathbf{A}$ be a symmetric matrix. Then:

$$\frac{\partial(\mathbf{a}'\mathbf{x})}{\partial \mathbf{x}} = \mathbf{a} \qquad (11.4.1a)$$

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} \qquad (11.4.1b)$$

Regression parameters are obtained through minimization of the quantity:

$$Q = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

The Least Squares Estimator $\mathbf{b}$

Consider the model:

$$\mathbf{Y} \sim \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Further assume that the design matrix $\mathbf{X}$ has full column rank with rank$(\mathbf{X}) = p$. Then the least squares estimator for $\boldsymbol{\beta}$ is given by:

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \qquad (11.4.2)$$

General Linear Model in Matrix Terms

In matrix terms, the multiple linear regression model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_{p-1}X_{i,p-1} + \varepsilon_i$$

is given by:

$$\mathbf{Y} \sim \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

- The matrix $\mathbf{X}$ is the **design matrix**.
- The vector $\boldsymbol{\beta}$ contains the true population regression coefficients.
- The response vector $\mathbf{Y}$ and errors $\boldsymbol{\varepsilon}$ are random.

The Hat Matrix $\mathbf{H}$

Suppose the design matrix $\mathbf{X}$ has full column rank with rank$(\mathbf{X}) = p$. Define $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, so $\hat{\mathbf{Y}} = \mathbf{H}\mathbf{Y}$. Then:

- $\mathbf{H}$ is symmetric. 11.6.1a
- $\mathbf{H}$ is idempotent. 11.6.1b
- rank$(\mathbf{H}) = p$. 11.6.1c

Distribution of Quadratic Forms

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix and $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with rank$(\boldsymbol{\Sigma}) = n$. Then:

- $\mathbf{Y}'\mathbf{A}\mathbf{Y} \sim \chi^2_{\text{rank}(\mathbf{A}),\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}}$

  if and only if $\mathbf{A}\boldsymbol{\Sigma}$ is idempotent. 12.1.3:
- $\mathbf{B}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{0}$ if and only if:
  - a) $\mathbf{Y}'\mathbf{A}\mathbf{Y} \perp \mathbf{B}\mathbf{Y}$, and 12.1.4a
  - b) $\mathbf{Y}'\mathbf{A}\mathbf{Y} \perp \mathbf{Y}'\mathbf{B}\mathbf{Y}$ 12.1.4b

The Normal Errors Regression Model

Under the normal errors model, $\varepsilon_i \sim iid\, N(0, \sigma^2)$. In matrix terms: $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$. The normal simple linear regression model is then:

$$\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$$

- $\mathbf{b} \sim N_p(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$ 11.5.2
- $\hat{\mathbf{Y}} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{H})$ 11.6.4
- $\mathbf{e} \sim N_n(\mathbf{0}, \sigma^2(\mathbf{I}_n - \mathbf{H}))$ 11.6.4

Properties of $\mathbf{b}, \hat{\mathbf{Y}}, \mathbf{e}$

- E$[\mathbf{b}] = \boldsymbol{\beta}$ i.e. $\mathbf{b}$ is unbiased. 11.5.1a
- Var$(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ 11.5.1b
- E$[\hat{\mathbf{Y}}] = \mathbf{X}\boldsymbol{\beta}$ 11.6.2a
- Var$(\hat{\mathbf{Y}}) = \sigma^2\mathbf{H}$ 11.6.2b
- E$[\mathbf{e}] = \mathbf{0}$ 11.6.3a
- Var$(\mathbf{e}) = \sigma^2(\mathbf{I}_n - \mathbf{H})$ 11.6.3b

Distribution Results for SSE and SSR

Let $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$. Then:

- E$(MSE) = \sigma^2$
- $\frac{SSE}{\sigma^2} \sim \chi^2_{n-p}$.
- E$(MSR) = \sigma^2 + \frac{1}{p-1}\boldsymbol{\beta}'\mathbf{X}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}\boldsymbol{\beta}$
- $\frac{SSR}{\sigma^2} \sim \chi^2_{p-1,\lambda}, \lambda = \frac{1}{\sigma^2}\boldsymbol{\beta}'\mathbf{X}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}\boldsymbol{\beta}$
- $SSR \perp SSE$