Matrix Arithmetic

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$
$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$
$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$$
$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})=\mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}$$

Idempotent

If
$$\mathbf{A}^2=\mathbf{A}$$
then $\mathbf{A}$ is called **idempotent**.

The property $\mathbf{A}^2=\mathbf{A}$ implies
$$\mathbf{A}^m=\mathbf{A}$$
for any positive integer m.

Determinant

For a 2×2 matrix: $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant is defined as:
$$|\mathbf{A}|=ad-bc$$
and the inverse is:
$$\mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Transpose and Trace of a Matrix

$$(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$$
$$(\mathbf{AB})'=\mathbf{B}'\mathbf{A}'$$
$$(\mathbf{ABC})'=\mathbf{C}'\mathbf{B}'\mathbf{A}'$$
$$\text{tr}(c\mathbf{A})=c\times\text{tr}(\mathbf{A})$$
$$\text{tr}(\mathbf{A}+\mathbf{B})=\text{tr}(\mathbf{A})+\text{tr}(\mathbf{B})$$
$$\text{tr}(\mathbf{A}')=\text{tr}(\mathbf{A})$$
$$\text{tr}(\mathbf{AB})=\text{tr}(\mathbf{BA})$$

Rank of a Matrix

The **rank** of an $r\times c$ matrix $\mathbf{A}$ is the number of linearly independent rows (or columns), rank$(\mathbf{A})$.

- rank$(\mathbf{A})\le\min(r,c)$
- rank$(\mathbf{A})=\min(r,c)\Rightarrow\mathbf{A}$ is of full rank.
- If $\mathbf{A}$ is an $n\times n$ matrix, then:
  rank$(\mathbf{A})=n\Leftrightarrow\mathbf{A}^{-1}$ exists $\Leftrightarrow|\mathbf{A}|\ne 0$
- rank$(\mathbf{A})=$ rank$(\mathbf{A}'\mathbf{A})=$ rank$(\mathbf{A}\mathbf{A}')$
- If $\mathbf{A}$ is idempotent, then rank$(\mathbf{A})=$ tr$(\mathbf{A})$

Expectation and Variance

Let $\mathbf{Y}$ be a random vector with mean $\boldsymbol{\mu}$. The **variance-covariance matrix** $\boldsymbol{\Sigma}$ is:
$$\text{Var}(\mathbf{Y})=\text{E}[(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})']=\boldsymbol{\Sigma}$$
Let $\mathbf{A}$ and $\mathbf{B}$ be nonrandom (constant) matrices.

a) $\text{E}(\mathbf{A}+\mathbf{BY})=\mathbf{A}+\mathbf{B}\boldsymbol{\mu}$ 11.1.1a

b) $\text{Var}(\mathbf{A}+\mathbf{BY})=\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'$ 11.1.1b

c) $\text{E}(\mathbf{Y}'\mathbf{AY})=\text{tr}(\mathbf{A}\boldsymbol{\Sigma})+\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ 12.1.1

Matrix Inverse

$$\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$$
$$\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A}$$
$$(\mathbf{A}')^{-1}=\left(\mathbf{A}^{-1}\right)'$$
$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$(\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$
If $\mathbf{A}^{-1}$ does not exist, we say $\mathbf{A}$ is **singular**.

The Conditional Inverse $\mathbf{A}^c$

Let $\mathbf{A}$ be a matrix. If a matrix $\mathbf{G}$ satisfies:
$$\mathbf{AGA}=\mathbf{A}$$
then $\mathbf{G}$ is called a **conditional inverse** of $\mathbf{A}$.

- If $\mathbf{A}$ is a square, nonsingular matrix, then the unique conditional inverse of $\mathbf{A}$ is $\mathbf{A}^{-1}$. Otherwise, $\mathbf{A}^c$ is not unique.
- rank$(\mathbf{A})=$ rank$(\mathbf{A}^c\mathbf{A})$
- $\mathbf{A}^c\mathbf{A}$ is idempotent, so rank$(\mathbf{A})=$ tr$(\mathbf{A}^c\mathbf{A})$

Quadratic Forms: Theorem 12.1.2

Suppose $\mathbf{Y}\sim N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$, and let $\mathbf{A}$ and $\mathbf{B}$ be constant matrices. Then

a) $\text{Var}(\mathbf{Y}'\mathbf{AY})=2\text{tr}(\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma})+4\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$

b) $\text{Cov}(\mathbf{Y},\mathbf{Y}'\mathbf{AY})=2\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$

c) $\text{Cov}(\mathbf{BY},\mathbf{Y}'\mathbf{AY})=2\mathbf{B}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$