## Interval Estimate

An **interval estimate** of a real-valued parameter $\theta$ is any pair of functions, $L(x_1, \dots, x_n)$ and $U(x_1, \dots, x_n)$, of a sample that satisfy $L(\mathbf{x}) \leq U(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$. If $\mathbf{X} = \mathbf{x}$ is observed, the inference $L(\mathbf{x}) \leq \theta \leq U(\mathbf{x})$ is made. The random interval $[L(\mathbf{X}), U(\mathbf{X})]$ is called an **interval estimator**.

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## Coverage Probability

For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter $\theta$, the **coverage probability** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the probability that the random interval $[L(\mathbf{X}), U(\mathbf{X})]$ covers the true parameter, $\theta$. In symbols, it is defined by either $P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$ or $P(\theta \in [L(\mathbf{X}), U(\mathbf{X})] | \theta)$.

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## Confidence Coefficient

For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter $\theta$, the **confidence coefficient** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the infimum of the coverage probabilities:

$$\inf_\theta P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

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## Pivotal Quantity

A random variable

$$Q(\mathbf{X}, \theta) = Q(X_1, \dots, X_n, \theta)$$

is a **pivotal quantity** (or **pivot**) if the distribution of $Q(\mathbf{X}, \theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x}|\theta)$, then $Q(\mathbf{X}, \theta)$ has the same distribution for all values of $\theta$.

## Confidence Interval

An interval estimator, together with a measure of confidence, is sometimes known as a **confidence interval**.

## Confidence Set

When working in general, we speak of confidence **sets**. A confidence set with confidence coefficient equal to $1 - \alpha$ is called a **$1 - \alpha$ confidence set.**

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## Pivoting a Continuous CDF

Let $T$ be a statistic with continuous cdf $F_T(t|\theta)$. Let $\alpha_1 + \alpha_2 = \alpha$ with $0 < \alpha < 1$ be fixed values. Suppose that for each $t \in \mathcal{T}$, the functions $\theta_L(t)$ and $\theta_U(t)$ can be defined as follows.

**i.** If $F_T(t|\theta)$ is a decreasing function of $\theta$ for each $t$, define $\theta_L(t)$ and $\theta_U(t)$ by

$$F_T(t|\theta_U(t)) = \alpha_1, \quad F_T(t|\theta_L(t)) = 1 - \alpha_2$$

**ii.** If $F_T(t|\theta)$ is an increasing function of $\theta$ for each $t$, define $\theta_L(t)$ and $\theta_U(t)$ by

$$F_T(t|\theta_U(t)) = 1 - \alpha_2, \quad F_T(t|\theta_L(t)) = \alpha_1$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for $\theta$.

## Pivoting a Discrete CDF

Let $T$ be a discrete statistic with cdf $F_T(t|\theta) = P(T \leq t|\theta)$. Let $\alpha_1 + \alpha_2 = \alpha$ with $0 < \alpha < 1$ be fixed values. Suppose that for each $t \in \mathcal{T}$, $\theta_L(t)$ and $\theta_U(t)$ can be defined as follows.

**i.** If $F_T(t|\theta)$ is a decreasing function of $\theta$ for each $t$, define $\theta_L(t)$ and $\theta_U(t)$ by

$$P(T \leq t|\theta_U(t)) = \alpha_1, P(T \geq t|\theta_L(t)) = \alpha_2$$

**ii.** If $F_T(t|\theta)$ is an increasing function of $\theta$ for each $t$, define $\theta_L(t)$ and $\theta_U(t)$ by

$$P(T \geq t|\theta_U(t)) = \alpha_1, P(T \leq t|\theta_L(t)) = \alpha_2$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for $\theta$.

## Inverting a Test Statistic

For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level $\alpha$ test of $H_0 : \theta = \theta_0$. For each $\mathbf{x} \in \mathcal{X}$, define a set $C(\mathbf{x})$ in the parameter space by

$$C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}$$

Then the random set $C(\mathbf{X})$ is a $1 - \alpha$ confidence set.

Conversely, let $C(\mathbf{X})$ be a $1 - \alpha$ confidence set. For any $\theta_0 \in \Theta$, define

$$A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$$

Then $A(\theta_0)$ is the acceptance region of a level $\alpha$ test of $H_0 : \theta = \theta_0$.