Hypothesis

A **hypothesis** is a statement about a population parameter.

Null/Alternate Hypotheses

The two complementary hypotheses in a hypothesis testing problem are called the **null hypothesis** and the **alternative hypothesis**. They are denoted by H_0 and H_1.

Hypothesis Test

A **hypothesis test** is a rule that specifies for which sample values H_0 is rejected (the **rejection region** R or **critical region**), and for which sample values H_0 is accepted as true (the **acceptance region**).

Power Function

The **power function** of a hypothesis test is the function of θ: $\beta(\theta) = P_\theta(\mathbf{X} \in R)$.
A test with power function $\beta(\theta)$ is **unbiased** if $\beta(\theta') \geq \beta(\theta'')$ for every $\theta' \in \Theta_0^c$ and $\theta'' \in \Theta_0$.

Size α/Level α Tests

For $0 \leq \alpha \leq 1$, a test with power function $\beta(\theta)$ is a **size** α **test** if: $\sup_{\theta\in\Theta_0} \beta(\theta) = \alpha$.
For $0 \leq \alpha \leq 1$, a test with power function $\beta(\theta)$ is a **level** α **test** if: $\sup_{\theta\in\Theta_0} \beta(\theta) \leq \alpha$.

Likelihood Ratio Test Statistic

To test $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_0^c$:

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{x})}{\sup_{\Theta} L(\theta|\mathbf{x})} = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})} = \frac{\sup_{\Theta_0} L^*(\theta|T(\mathbf{x}))}{\sup_{\Theta} L^*(\theta|T(\mathbf{x}))}$$

where $T(\mathbf{X})$ is a sufficient statistic for θ.

Likelihod Ratio Test

A **likelihood ratio test** (LRT) is any test that has a rejection region of the form:

$$\{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$$

where c is any number satisfying $0 \leq c \leq 1$.

UMP Class $\mathcal{C}$ Test

A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a **uniformly most powerful (UMP) class** $\mathcal{C}$ **test** if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.

Neyman-Pearson Lemma

If a test of $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ satisfies

$$(i) \quad \mathbf{x} \in R \quad \text{if} \quad f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0), \text{ and}$$
$$\mathbf{x} \in R^c \quad \text{if} \quad f(\mathbf{x}|\theta_1) < kf(\mathbf{x}|\theta_0)$$

for some $k \geq 0$, and (ii) $\alpha = P_{\theta_0}(\mathbf{X} \in R)$:
a. Any test satisfying (i, ii) is UMP level α.
b. If $k > 0$ for any test satisfying (i, ii), all UMP level α satisfy (ii), and (i) except perhaps on A: $P_{\theta_0}(\mathbf{X} \in A) = P_{\theta_1}(\mathbf{X} \in A) = 0$.

Neyman-Pearson and Sufficiency

A test based on $T(\mathbf{X}) \sim g(t|\theta_i)$, sufficient for θ, rejection region S, is UMP level α if:

$$t \in S \quad \text{if} \quad g(t|\theta_1) > kg(t|\theta_0), \text{ and}$$
$$t \in S^c \quad \text{if} \quad g(t|\theta_1) < kg(t|\theta_0)$$

for some $k \geq 0$, where $\alpha = P_{\theta_0}(T \in S)$

Error Types

Type I Error $\theta \in \Theta_0$ but H_0 is incorrectly rejected.

Type II Error $\theta \in \Theta_0^c$ but H_0 is incorrectly accepted.

Monotone Likelihood Ratio

A family $\{g(t|\theta) : \theta \in \Theta\}$ has a **monotone likelihood ratio** (MLR) if, for every $\theta_2 > \theta_1, g(t|\theta_2)/g(t|\theta_1)$ is a monotone function of t on $\{t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\}$. Note that $c/0$ is defined as ∞ if $0 < c$.

Karlin-Rubin Theorem

To test $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$. If:
- T is a sufficient statistic for θ;
- $\{g(t|\theta) : \theta \in \Theta\}$ of T has an MLR
Then for any t_0, the test that rejects H_0 if and only if $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.

p-Value

A **p-value** $p(\mathbf{X})$ is a test statistic satisfying $0 \leq p(\mathbf{x}) \leq 1$ for every point $\mathbf{x}$. Small values of $p(\mathbf{X})$ give evidence that H_1 is true.
A p-value is **valid** if, for every $\theta \in \Theta_0$ and every $0 \leq \alpha \leq 1$, $P_\theta(p(\mathbf{X}) \leq \alpha) \leq \alpha$

Constructing a Valid p-Value

Let $W(\mathbf{X})$ be a test statistic such that large values of W give evidence that H_1 is true. For each sample point $\mathbf{x}$, define

$$p(\mathbf{x}) = \sup_{\theta\in\Theta_0} P_\theta(W(\mathbf{X}) \geq W(\mathbf{x}))$$

Then $p(\mathbf{X})$ is a valid p-value.