Converges Almost Surely: $X_n \stackrel{a.s.}{\to} X$

A sequence of random variables $X_1, X_2, \ldots$, **converges almost surely** to a random variable X if, for every $\epsilon > 0$,

$$P\left(\lim_{n\to\infty} |X_n - X| < \epsilon\right) = 1$$

$\Rightarrow$

Converges in Probability: $X_n \stackrel{P}{\to} X$

A sequence of random variables $X_1, X_2, \ldots$, **converges in probability** to a random variable X if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

$\stackrel{X=\mu}{\Leftarrow}$

$\Rightarrow$

Converges in Distribution: $X_n \stackrel{D}{\to} X$

A sequence of random variables $X_1, X_2, \ldots$, **converges in distribution** to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points x where $F_X(x)$ is continuous.

$\downarrow$

Convergence of $h(X_i)$ to $h(X)$

Suppose that $X_1, X_2, \ldots$ converges in probability to a random variable X and that h is a continuous function. Then

$$h(X_1), h(X_2), \ldots \stackrel{P}{\to} h(X)$$

$\downarrow$

Slutsky's Theorem

If $X_n \to X$ in distribution and $Y_n \to a$, a constant, in probability, then

- $Y_n X_n \stackrel{D}{\to} aX$
- $X_n + Y_n \stackrel{D}{\to} X + a$

Strong Law of Large Numbers

For every $\epsilon > 0$

$$P\left(\lim_{n\to\infty} |\bar{X}_n - \mu| < \epsilon\right) = 1$$

i.e. $\bar{X}_n$ converges almost surely to μ

Weak Law of Large Numbers

For every $\epsilon > 0$

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \epsilon) = 1$$

i.e. $\bar{X}_n$ converges in probability to μ

Central Limit Theorem, Var$[X_i] > 0$

$G_n(x)$ is cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$. For any x, $-\infty < x < \infty$

$$\lim_{n\to\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$\nwarrow$ $\uparrow$ $\nearrow$

- $X_1, X_2, \ldots$ iid random variables
- E$[X_i] = \mu$
- Var$[X_i] = \sigma^2 < \infty$
- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$