Consistent Sequence of Estimators

A sequence of estimators $W_n = W_n(X_1, \ldots, X_n)$ is a **consistent sequence of estimators** of the parameter θ if, for every $\epsilon > 0$ and every $\theta \in \Theta$,

$$\lim_{n\to\infty} P_\theta(|W_n - \theta| < \epsilon) = 1$$

Sufficient Conditions for Consistency

If W_n is a sequence of estimators of a parameter θ satisfying

i. $\lim_{n\to\infty} \mathrm{Var}_\theta W_n = 0$,

ii. $\lim_{n\to\infty} \mathrm{Bias}_\theta W_n = 0$.

for every $\theta \in \Theta$, then W_n is a consistent sequence of estimators of θ.

Consistency of MLEs

Let $X_1, X_2, \ldots$, be iid $f(x|\theta)$, $\hat{\theta}$ the MLE of θ, and $\tau(\theta)$ a continuous function of θ. Under the regularity conditions in Miscellanea 10.6.2 on $f(x|\theta)$, for every $\epsilon > 0$ and every $\theta \in \Theta$,

$$\lim_{n\to\infty} P_\theta(|\tau(\hat{\theta}) - \tau(\theta)| \geq \epsilon) = 0$$

i.e. $\tau(\hat{\theta})$ is a consistent estimator of $\tau(\theta)$.

Asymptotic Variance

For an estimator T_n, suppose that

$$k_n(T_n - \tau(\theta)) \to \mathsf{n}(0, \sigma^2)$$

in distribution. The parameter σ^2 is called the **asymptotic variance** or *variance of the limit distribution* of T_n.

Asymptotic Distrib. of Simple LRT

For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, suppose $X_1, \ldots, X_n$ are iid $f(x|\theta)$, $\hat{\theta}$ is the MLE of θ, and $f(x|\theta)$ satisfies the regularity conditions in Miscellanea 10.6.2. Then under H_0, as $n \to \infty$,

$$-2\log\lambda(\mathbf{X}) \to \chi^2_1 \text{ in distribution}$$

where χ^2_1 is a χ^2 random variable with 1 degree of freedom.

Wald Test

In general, a **Wald test** is a test based on a statistic of the form

$$Z_n = \frac{W_n - \theta_0}{S_n}$$

where θ_0 is a hypothesized value of the parameter θ, W_n is an estimator of θ, and S_n is a standard error for W_n, an estimate of the standard deviation of W_n.

Asymptotically Efficient

A sequence of estimators W_n is **asymptotically efficient** for a parameter $\tau(\theta)$ if

$$\sqrt{n}[W_n - \tau(\theta)] \to \mathsf{n}[0, v(\theta)]$$

in distribution and

$$v(\theta) = \frac{[\tau'(\theta)]^2}{\mathsf{E}_\theta\left(\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right)}$$

Asymptotic Efficiency of MLEs

Let $X_1, X_2, \ldots$, be iid $f(x|\theta)$, let $\hat{\theta}$ denote the MLE of θ, and let $\tau(\theta)$ be a continuous function of θ. Under the regularity conditions in Miscellanea 10.6.2 on $f(x|\theta)$ and, hence, $L(\theta|\mathbf{x})$,

$$\sqrt{n}[\tau(\hat{\theta}) - \tau(\theta)] \to \mathsf{n}[0, v(\theta)]$$

That is, $\tau(\hat{\theta})$ is a consistent and asymptotically efficient estimator of $\tau(\theta)$.

Approximate Variance of an MLE

Suppose that $X_1, \ldots, X_n$ are iid $f(x|\theta)$, $\hat{\theta}$ is the MLE of θ, and $I_n(\theta) = \mathsf{E}_\theta\left(\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{X})\right)^2$ is the information number of the sample. The variance of $h(\hat{\theta})$ can be approximated by:

$$\mathrm{Var}(h(\hat{\theta})|\theta) \approx \frac{[h'(\theta)]^2}{I_n(\theta)} \approx \frac{[h'(\theta)]^2\big|_{\theta=\hat{\theta}}}{-\frac{\partial^2}{\partial\theta^2}\log L(\theta|\mathbf{X})\big|_{\theta=\hat{\theta}}}$$