The Deligne–Simpson problem for connections on $\mathbb{G}_m$ with a maximally ramified singularity

A natural question in the theory of systems of meromorphic differential equations (or equivalently, meromorphic connections) on $\mathbb{P}^1$ is whether there exists a global connection with specified local behavior at a collection of singular points. The Deligne-Simpson problem is concerned with a variant of this question. The classical additive Deligne–Simpson problem is the existence problem for Fuchsian connections whose residues at the singular points are contained in specified adjoint orbits. Crawley-Boevey solved this problem in 2003 by reinterpreting it in terms of quiver varieties. A more general version of the problem, solved by Hiroe in 2017, allows additional unramified irregular singularities. We apply the theory of fundamental and regular strata due to Bremer and Sage to formulate a version of the Deligne–Simpson problem in which certain ramified singularities are allowed. These ramified singular points are called toral singularities; they are singularities whose leading term with respect to a Moy–Prasad filtration is regular semisimple. We solve this problem in a special case that plays an important role in recent work on the geometric Langlands program: connections on $\mathbb{G}_m$ with a maximally ramified singularity at 0 and possibly an additional regular singular point at infinity. We also give a complete characterization of all such connections that are rigid, under the additional hypothesis of unipotent monodromy at infinity.