MTH 620: 2020-04-14 lecture

Alexandru Chirvasitu

1 Tensor products

You are doubtless familiar with the fact that for topological spaces X the cohomology

$$H^*(X) := \bigoplus_{n\geq 0} H^n(X)$$

is a *graded ring*, in the sense that it comes equipped with a multiplication

$$H^m(X) \otimes H^n(X) \to H^{m+n}(X),\ m,n \in \mathbb{Z}_{\geq 0}.$$

Since, as discussed before, the cohomology of a group G is the singular cohomology of its classifying space BG, we must have the same type of structure on

$$H^*(G) := \bigoplus_{n\geq 0} H^n(G).$$

This multiplicative structure (in a somewhat more general setup) will be the focus of this lecture. We will have to refer back to the previous lecture [1] quite a bit.

1.1 Tensor products of complexes

This is a brief recollection of a construction discussed in class. We denote by C^* and D^* two cochain complexes with differentials

$$\partial_C : C^n \to C^{n+1} \quad \text{and} \quad \partial_D : D^n \to D^{n+1}$$

respectively. In earlier lectures we encountered

Definition 1.1 The *tensor product* $C^* \otimes D^*$ is the cochain complex E^* with terms

$$E^n := \bigoplus_{p+q=n} C^p \otimes C^q,$$

whose differential $\partial_E : E^n \to E^{n+1}$, restricted to the summand $C^p \otimes D^q$ of E^n, is given by

$$\partial_E(x \otimes y) := \partial_C(x) \otimes y + (-1)^p x \otimes \partial_D(y).$$ ♦

Remark 1.2 The sign convention in Definition 1.1 is very common in homological algebra, and you might find it referred to as *sign rule*. The mnemonic for it is that you should think of $x \in C^p$ as having degree p and of the symbol ∂ as having "degree 1", and in general, you pick up a $(-1)^{mn}$ sign whenever you flip two symbols of degrees m and n. Since you passed a ∂ (of degree 1) over an x (of degree p), that's the $(-1)^p$. ♦

We are interested in cohomology though, hence the relevance of the following

Proposition 1.3 *Let (C^*, ∂_C) and (D^*, ∂_D) be two cochain complexes. Then, there are well-defined maps*

$$H^p(C^*) \otimes H^q(D^*) \to H^{p+q}(C^* \otimes D^*)$$

sending

$$(\textit{class of } x \in C^p) \otimes (\textit{class of } y \in D^p)$$

to

$$\textit{class of } x \otimes y \in C^p \otimes D^q \subset (C^* \otimes D^*)^{p+q}.$$

Problem 1 *Prove Proposition 1.3.*

You'll have to prove a number of things, like say the fact that $x \otimes y$ is a cocycle whenever both x and y are, and also that the image in cohomology only depends on the cohomology classes of x and y.

1.2 Coalgebras, bialgebras, and tensor products of G-modules

The category ${}_G\mathrm{Mod}$ is of course a category of modules by definition, but it has more structure because of the special nature of the group ring $\mathbb{Z}G$.

In general, for any two rings R and S and modules $M \in {}_R\mathrm{Mod}$, $N \in {}_S\mathrm{Mod}$ the tensor product $M \otimes N$ (over $\mathbb{Z}$) acquires a natural module structure over $R \otimes S$: the action of the latter ring is given by

$$(r \otimes s)(m \otimes n) = rm \otimes sn,\ r \in R,\ s \in S,\ m \in M,\ n \in N,$$

as you would have expected. This is often referred to as the *external tensor product* of M and N.

In particular, this applies to $R = S = \mathbb{Z}G$ and two modules $M, N \in {}_G\mathrm{Mod}$ to make $M \otimes N$ into a module over

$$\mathbb{Z}G \otimes \mathbb{Z}G \cong \mathbb{Z}[G \times G].$$

This is not the end of the story though: $\mathbb{Z}G$ comes equipped with a ring morphism

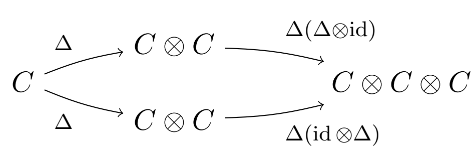
$$\mathbb{Z}G \ni g \mapsto g \otimes g \in \mathbb{Z}G \otimes \mathbb{Z}G.$$

This allows us to pull back the $\mathbb{Z}G \otimes \mathbb{Z}G$-module structure on $M \otimes N$ back to $\mathbb{Z}G$ by scalar restriction, and hence make $M \otimes N$ into a G-module. In summary, the action of G on $M \otimes N$ is simply given by

$$g(m \otimes n) = gm \otimes gn.$$

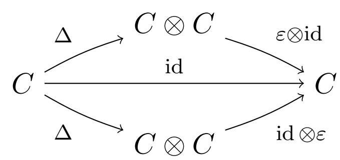
As an aside, what makes $\mathbb{Z}G$ special is that it is not simply an algebra but a *bi*algebra. First, consider the following dualization of the notion of a ring.

Definition 1.4 A *coring* C is an abelian group equipped with a map $\Delta : C \to C \otimes C$ (the *comultiplication*, or *coproduct*) that is *coassociative*, in the sense that



commutes.

C is *counital* if it is further equipped with an abelian group morphism $\varepsilon : C \to \mathbb{Z}$ (the *counit*) such that



commutes.

As for rings, we occasionally use the term *coalgebra* instead of *coring*, especially when working over a field instead of $\mathbb{Z}$. ♦

Unless specified otherwise, we work with counital corings/coalgebras (as was the case for rings, which for us are assumed unital).

Remark 1.5 Note that Definition 1.4 is precisely parallel to the definition of a ring, except that all arrows are reversed: you have a comultiplication map $C \to C \otimes C$ instead of multiplication $R \otimes R \to R$, a counit $C \to \mathbb{Z}$ instead of a unit $\mathbb{Z} \to R$, etc. ♦

To get back to a term mentioned in passing above:

Definition 1.6 A *(unital and counital) bialgebra* is an abelian group B equipped with

- an algebra structure $m : B \otimes B \to B$, $\eta : \mathbb{Z} \to B$ and
- a coalgebra structure $\Delta : B \to B \otimes B$, $\varepsilon : B \to \mathbb{Z}$

that are compatible in the sense that Δ and ε are both morphisms of unital algebras. ♦

Definition 1.6 seems somewhat unbalanced: we asked that the coalgebra structure maps be algebra morphisms but not the other way around. It turns out that this was unnecessary, as it would lead to exactly the same notion:

Problem 2 *Let B be an algebra and coalgebra as in the first part of Definition 1.6. Prove that the two conditions below are equivalent:*

(a) Δ and ε are morphisms of unital algebras;

(b) m and η are morphisms of counital coalgebras.

Finally, after all of this, we can get to the point (of this bialgebra-themed aside):

Proposition 1.7 *For every group G, the group algebra $\mathbb{Z}G$ becomes a bialgebra when equipped with the comultiplication*

$$\Delta : g \mapsto g \otimes g \in \mathbb{Z}G \otimes \mathbb{Z}G$$

and the counit

$$\varepsilon : g \mapsto \delta_{g,1} := \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, the G-module $M \otimes N$ (for $M, N \in {}_G\mathrm{Mod}$) can be recovered as an application of the general remark that for a bialgebra B with B-modules M and N, $M \otimes N$ is again a B-module by

- first making it into a $B \otimes B$-module via the external tensor product construction discussed above;
- then restricting scalars back to B via the comultiplication $B \to B \otimes B$.

2 Multiplicative structure on $H^*(G)$

Let G be a group. We will construct "multiplication" morphisms

$$H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N),\ p, q \in \mathbb{Z}_{\geq 0},$$

functorial in $M, N \in {}_G\mathrm{Mod}$, where $M \otimes N \in {}_G\mathrm{Mod}$ is the tensor product module constructed in §1.2.

We proceed as follows. First, consider the canonical contractible resolutions

$$0 \to M \to C^0 \to C^1 \to \cdots$$

and

$$0 \to N \to D^0 \to D^1 \to \cdots$$

of M and N introduced in the previous lecture ([1, Proposition 3.2]). Because they are contractible, so is the complex

$$0 \to M \otimes N \to E^0 \to E^1 \to \cdots ,$$

where

- $E^* := C^* \otimes D^*$ is the tensor product cochain complex, as discussed in §1.1;
- all tensor products of G-modules are again considered as G-modules, as in §1.2.

In other words, E^* is a *resolution* of $M \otimes N$. This means that for every *injective* resolution

$$0 \to M \otimes N \to I^0 \to \cdots$$

we have

- a morphism
$$E^* \to I^* \tag{2-1}$$
of complexes that extends the identity on $M \otimes N$;
- which is unique up to homotopy.

We now have all of the ingredients, and can start putting them together.

(1) First, note that for any two G-modules X and Y we have a canonical abelian group morphism

$$X^G \otimes Y^G \to (X \otimes Y)^G$$

that simply sends $x \otimes y$ to $x \otimes y$ again, for $x \in X^G$ and $y \in Y^G$. Assembling together such maps in each degree will give us a morphism

$$C^{*G} \otimes D^{*G} \to (C^* \otimes D^*)^G = E^{*G}$$

of cochain complexes (of abelian groups). According to Proposition 1.3, this then induces morphisms

$$H^p(G, M) \otimes H^q(G, N) \cong H^p(C^{*G}) \otimes H^q(D^{*G}) \to H^{p+q}(E^{*G}). \tag{2-2}$$

Note that we have used [1, Theorem 1.2] to conclude that the cohomology of M can be computed via the acyclic (though in general not injective) resolution C^*, and similarly for N.

(2) Apply the fixed-point functor $(-)^G$ to (2-1) and then take cohomology to get maps

$$H^{p+q}(E^{*G}) \to H^{p+q}(I^{*G}) \cong H^{p+q}(G, M \otimes N). \tag{2-3}$$

(3) Finally, compose (2-2) and (2-3) to get the sought-after morphisms

$$H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N). \tag{2-4}$$

Definition 2.1 Let G be a group and $M, N \in {}_G\mathrm{Mod}$ two G-modules. The *cup product*

$$\cup : H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N)$$

is the map (2-4) constructed above. ♦

[2, Theorem, p.585] offers a good summary of the properties of these maps. Ignore the hats on top of the cohomology groups; they denote *Tate cohomology*, which we haven't discussed; it specializes back to ordinary cohomology for positive integer degrees.

The construction of cup products given above is well motivated conceptually, but not the most straightforward definition to work with.. We will have more explicit descriptions of products later, in future lectures.

2.1 Cohomology algebras

If you recall, this lecture was initially motivated by the fact that the (integer-valued, say) singular cohomology $H^*(X)$ of a space forms an algebra. This is the case for the cohomology $H^*(G, M)$ of a *single* module M (as opposed to Definition 2.1, which requires two modules), but we need extra structure on the module. The relevant concept is

Definition 2.2 Let G be a group. A *G-algebra* is a G-module A that is also an algebra (i.e. ring) such that

- the unit $1 \in A$ is fixed by G, i.e. $1 \in A^G$;
- the multiplication $A \otimes A \to A$ is a G-module morphism.

This latter condition simply says that

$$g(ab) = (ga)(gb),\ \forall a, b \in A,\ \forall g \in G.$$

We say that A is a *commutative G-algebra* if it is a G-algebra that is commutative as an algebra.♦

For a G-algebra A we have the cup product

$$\cup : H^p(G, A) \otimes H^q(G, A) \to H^{p+q}(G, A \otimes A)$$

from Definition 2.1, which we can follow up with the map

$$H^{p+q}(G, A \otimes A) \to H^{p+q}(G, A)$$

obtained by applying the functor $H^{p+q}(G, -)$ to the G-module multiplication morphism $A \otimes A \to A$. See also the discussion immediately preceding [2, §9.8].

We summarize the discussion in

Proposition 2.3 *Let G be a group and A a G-algebra.*

(a) Equipped with the multiplication morphisms

$$H^p(G,A) \otimes H^q(G,A) \to H^{p+q}(G,A)$$

from above the direct sum

$$H^*(G,A) := \bigoplus_{n\ge 0} H^n(G,A)$$

is a graded algebra. In particular, the multiplication introduced above is associative, and the class of $1 \in A^G = H^0(G,A)$ is a unit for the multiplication.

(b) If furthermore A is commutative then $H^*(G,A)$ is anticommutative (or skew-commutative or super-commutative) in the sense that

$$yx = (-1)^{pq}xy,\ \forall x \in H^p(G,A),\ y \in H^q(G,A). \tag{2-5}$$

As is customary with graded algebras, call an element $x \in H^*(G,A)$ *homogeneous* if it belongs to one of the summands $H^p(G,A)$, and in that case refer to p as the *degree* of x. We also fix the notation $|x| = p$ for the degree. With that in place, (2-5) says

$$yx = (-1)^{|x||y|}xy,\ \forall \text{ homogeneous } x,y \in H^*(G,A).$$

3 A last problem

Very briefly, I mentioned above that for two G-modules $M,N \in {}_G\mathrm{Mod}$ we have a canonical abelian group morphism

$$M^G \otimes N^G \to (M \otimes N)^G. \tag{3-1}$$

This problem has you explore that map just a little bit:

Problem 3 *Is the map (3-1) always an isomorphism? Prove the claim or provide a counterexample.*

References

[1] Alexandru Chirvasitu. http://www.acsu.buffalo.edu/~achirvas/Math620_Spring2020/lec-2020-04-07.pdf.

[2] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.

Department of Mathematics, University at Buffalo, Buffalo, NY 14260-2900, USA
E-mail address: achirvas@buffalo.edu