MTH 620: 2020-04-07 lecture

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1 Acyclic resolutions

This will be a bit of a detour, revisiting general derived-functor theory before we circle back to group (co)homology. First, let's make sense of the title of the present section.

Definition 1.1 Let R be a ring and $F : {}_R\mathrm{Mod} \to \mathrm{Ab}$ a left exact functor. An object $X \in {}_R\mathrm{Mod}$ is F-*acyclic* (or just plain 'acyclic' when F is understood) if all right derived functors $R^iF(X)$, $i \ge 1$ vanish on X.

Similarly, for *right* exact F we call X F-acyclic if all *left* derived functors $L_iF(X)$, $i \ge 1$ vanish.⧫

This is fairly standard terminology; see e.g. [1, Definition, p.358, p.368 or p.379].

We'll specialize mostly to left exact (and hence right derived) functors. The usefulness of the concept for us stems from the fact that there's a general phenomenon, whereby computing the right derived functors $R^iF(X)$ (for an object X) doesn't actually require you resolve X by *injectives*, but rather it's enough to resolve by F-acyclic objects.

You can see a particular case of this in [1, Theorem 7.5], for Tor: there, the point is that in order to compute $\mathrm{Tor}_i(X, Y)$, it's enough to use a *flat* (rather than projective) resolution of either X or Y. I will state the general principle here (in its right-derived-functor incarnation), and then leave it as homework to prove it.

Theorem 1.2 *Let R be a ring, $F : {}_R\mathrm{Mod} \to \mathrm{Ab}$ a left exact functor, $X \in {}_R\mathrm{Mod}$ an object and*

$$0 \to X \to A_0 \to A_1 \to \cdots \tag{1-1}$$

a resolution of X (i.e. exact sequence) with all A_i F-acyclic. Then, the i^{th} derived functor $R^iF(X)$ can be computed as the i^{th} cohomology of the cochain complex

$$F(A_*) := 0 \to F(A_0) \to F(A_1) \to \cdots$$

of abelian groups.

In short, acyclic resolutions are just as good as injective ones for the purpose of computing derived functors. Needless to say, there is a version for right exact (and their left derived) functors that should be trivial to state at this point.

This brings us to

Problem 1 *Prove Theorem 1.2.*

The ensuing discussion is meant to get you started on Problem 1. My suggestion is you approach this by induction on i, the case $i = 0$ being easy (using nothing but the left exactness of F). Now, for $i \ge 1$, consider the short exact sequence that starts off (1-1):

$$0 \to X \to A_0 \to C \to 0,$$

where C is simply the cokernel of the embedding $X \to A_0$. You then get a long exact derived-functor sequence

$$\cdots \to R^{i-1}F(A_0) \to R^{i-1}F(C) \to R^iF(X) \to R^iF(A_0) \to \cdots .$$

If $i \ge 2$ the acyclicity of A_0 means that the two extreme terms displayed above vanish, so

$$R^{i-1}F(C) \cong R^iF(X).$$

This means that you can replace i with $i-1$, X with C, (1-1) with

$$0 \to C \to A_1 \to A_2 \to \cdots$$

and proceed by induction. So you're left having to prove the claim for $i = 1$, which I'll let you sort out.

2 Change of rings / groups and derived functors

The following observation should be fairly simple, given the way we defined derived functors (via projective or injective resolutions).

Proposition 2.1 *Let R and S be rings, and $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ an exact functor.*

(a) If F preserves injectivity and $G : {}_S\mathrm{Mod} \to \mathrm{Ab}$ is left exact, then we have a natural isomorphism

$$R^i(G \circ F) \cong (R^iG) \circ F.$$

for all $i \in \mathbb{Z}_{\ge 0}$.

(b) Dually, if F preserves projectivity and $G : {}_S\mathrm{Mod} \to \mathrm{Ab}$ is right exact, then we have a natural isomorphism

$$L_i(G \circ F) \cong (L_iG) \circ F.$$

for all $i \in \mathbb{Z}_{\ge 0}$. ■

We'll specialize this to functors resulting from ring (and eventually group) morphisms. To that end, let $R \to S$ be a ring homomorphism. We are interested in the following application of Proposition 2.1.

Problem 2 *Suppose S is projective as a left R-module.*

(a) Let M be a right R-module and N a left S-module. Show that we have isomorphisms

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^S(M \otimes_R S, N),$$

where on the left N is regarded as an R-module via scalar restriction.

(b) Let M be a left S-module and N a left R-module. Show that we have isomorphisms

$$\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_S(M, {}_R\mathrm{Hom}(S, N)),$$

where on the left M is regarded as an R-module via scalar restriction.

You'll want to show that you can take the functor F from Proposition 2.1 to be either

$$-\otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S \tag{2-1}$$

or

$${}_R\mathrm{Hom}(S,-) : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}, \tag{2-2}$$

and the hypotheses of Proposition 2.1 will be satisfied (you'll have to decide what G is in each case). For projectivity / injectivity preservation, you want

- For any ring morphism $R \to S$ the functor (2-1) turns projective modules into projective modules.
- Dually, (2-2) preserves injectivity.

In turn, you might want to use the fact that these functors are left and right adjoints respectively to scalar restriction from S-modules to R-modules, and scalar restriction is exact.

Finally, we can turn to groups. As an immediate consequence of Problem 2 we now have

Corollary 2.2 *Let $H \le G$ be a subgroup. Then, for every H-module M and arbitrary $i \in \mathbb{Z}_{\ge 0}$ we have*

$$H_i(H,M) \cong H_i(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M)$$

and

$$H^i(H,M) \cong H^i(G, {}_H\mathrm{Hom}(\mathbb{Z}G, M)).$$

■

This is *Shapiro's Lemma*, which you can also find as [1, Proposition 9.76]. It uses Problem 2 and the fact that given a group inclusion $H \le G$ the group algebra $\mathbb{Z}G$ is projective (indeed, even free, as seen in class) over $\mathbb{Z}H$.

In particular, taking H to be trivial in Corollary 2.2 we obtain

Corollary 2.3 *Let G be a group and A an abelian group. Then,*

(a) The G-module $\mathbb{Z}G \otimes A$ has trivial higher homology H_i, $i \ge 1$.

(b) The G-module $\mathrm{Hom}(\mathbb{Z}G, A)$ has trivial higher cohomology H^i, $i \ge 1$.

■

Modules of this type are important enough to warrant special terminology (see [1, Definition, p.561]):

Definition 2.4 Let G be a group.

An *induced* G-module is one of the form $\mathbb{Z}G \otimes A$, where A is an abelian group.

A *coinduced* G-module is one of the form $\mathrm{Hom}(\mathbb{Z}G, A)$, where A is an abelian group. ♦

Now that we have the language, we can restate Corollary 2.3 as saying that

(a) Induced G-modules are acyclic for the functor $(-)_G$ of G-coinvariants.

(b) Coinduced G-modules are acyclic for the functor $(-)^G$ of G-coinvariants.

3 Assembling the pieces

We're working our way up to a conclusion: Section 1 is about using acyclic resolutions to compute derived functors, while Section 2 provides a wealth of acyclic G-modules.

Let M be a G-module. You can then also consider M to be a plain abelian group (forgetting the G-action), and construct the associated coinduced module $\mathrm{Hom}(\mathbb{Z}G, M)$ (the Hom is over $\mathbb{Z}$). Now consider the map ψ_M sending an element $m \in M$ to the function $G \to M$ defined by

$$\psi_M(m)(g) := gm.$$

Identifying functions $G \to M$ to the coinduced G-module $\mathrm{Hom}(\mathbb{Z}G, M)$, this gives us a map

$$\psi_M : M \to \mathrm{Hom}(\mathbb{Z}G, M).$$

Problem 3 *Show that ψ_M is in fact a G-module embedding, and that it is functorial in $M \in {}_G\mathrm{Mod}$.*

Prove also that the embedding $\psi_M : M \to \mathrm{Hom}(\mathbb{Z}G, M)$ splits as an abelian group map (not necessarily as a G-module map!).

Remark 3.1 As everywhere in the discussion above, the coinduced module $\mathrm{Hom}(\mathbb{Z}G, M)$ is a G-module via *right* multiplication on the domain $\mathbb{Z}G$:

$$(gf)(x) := f(xg)$$

for all $g \in G$, $x \in \mathbb{Z}G$ and $f \in \mathrm{Hom}(\mathbb{Z}G, M)$. ♦

Problem 3 allows us to construct a canonical, functorial acyclic resolution for every G-module M:

$$0 \to M \to A_0 \to A_1 \to \cdots,$$

where

- $M \to A_0$ is the canonical embedding from Problem 3;
- which fits into a short exact sequence

  $$0 \to M \to A_0 \to C \to 0$$

  you can then follow up with another canonical embedding

  $$C \to A_1 := \mathrm{Hom}(\mathbb{Z}G, C);$$

- fitting into another short exact sequence

  $$0 \to C \to A_1 \to D \to 0$$

  and thus giving rise to another canonical embedding

  $$D \to A_2 := \mathrm{Hom}(\mathbb{Z}G, D);$$

- etc; continue this process recursively.

Furthermore, the fact that canonical embeddings

$$M \to \mathrm{Hom}(\mathbb{Z}G, M)$$

split over $\mathbb{Z}$ (as claimed by Problem 3) implies

Proposition 3.2 *For every $M \in {}_G\mathrm{Mod}$ the canonical acyclic resolution by coinduced modules constructed above is contractible.*

Let me remind you from class that 'contractible', for a complex, means that the identity morphism of the complex is chain-homotopic to the zero map.

This is excellent news: recall that the bar resolution of the trivial G-module $\mathbb{Z}$ was convenient, among other things, because it was contractible in the category Ab. That gave us a very useful *projective* resolution for computing

$$H^i(G, M) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M). \tag{3-1}$$

Now, on the other hand, we have contractible resolutions at the "other end" of (3-1), i.e. for M rather than $\mathbb{Z}$; furthermore, those resolutions are natural in M. This came at the cost of replacing *injectivity* with the weaker property of *acyclicity*.

References

[1] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.

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