MTH 620: 2020-03-31 lecture

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1 δ-functors revisited (briefly)

The previous lecture focused on the cohomology side of the change-of-groups picture. Here we briefly dualize the discussion, because there are some interesting applications specific to homology. [2, §9.6] will be a good reference for much of the new material.

The pertinent concept here is that of a *homological* (as opposed to cohomological) δ-functor. I will not reprise the definition, since

- it is a direct dualization of the one in the preceding lecture, and
- it appears as [2, Definition, p.359].

I will, however, recall (the homological version of) universality (see [1, Definition 1.2]):

Definition 1.1 A homological δ-functor $(F_i)_{i\ge 0}$ from ${}_R\mathrm{Mod}$ to Ab is *universal* if for every homological δ-functor $(F'_i)_i$, every natural transformation $\eta_0 : F'_0 \to F_0$ extends uniquely to a morphism

$$\eta_i : F'_i \to F_i$$

of δ-functors.

$(F_i)_i$ is *effaceable* if for every object X there is an epimorphism $P \to X$ such that $F_iP = 0$ for all $i \ge 1$. ♦

Just as in [1, Theorem 1.3], we have

Theorem 1.2 *Effaceable homological δ-functors are universal.* ■

And there's an analogue for [1, Corollary 1.4] too:

Corollary 1.3 *For every right exact functor $F : {}_R\mathrm{Mod} \to \mathrm{Ab}$ the left derived functors $(L_iF)_{i\in\mathbb{Z}_{\ge 0}}$ constitute an effaceable and hence universal homological δ-functor.* ■

2 Group change in homology

2.1 Corestriction

As before, we consider group morphisms $H \to G$, and we again consider G-modules M as H-modules via "scalar restriction" along the ring morphism $\mathbb{Z}H \to \mathbb{Z}G$. Recall that the coinvariant group M_H is by definition

$$M/(h-1,\ h \in H),$$

i.e. the largest quotient of M on which H acts trivially. Since H acts by restricting the G-action, the sum of the ranges of $h-1$, $h \in H$ (with $h-1$ regarded as operators on M) is contained in the sum of the ranges of $g-1$, $g \in G$. This means that we have a canonical morphism

$$M_H = M/(h-1,\ h \in H) \to M/(f-1,\ g \in G) = M_G.$$

This is clearly functorial in $M \in {}_G\mathrm{Mod}$, so we get a natural transformation

$$\begin{tikzcd} {}_G\mathrm{Mod} \arrow[r, bend left, "H_0(H,-)"] \arrow[r, bend right, "H_0(G,-)"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Ab} \end{tikzcd} \tag{2-1}$$

By the universality of the homological δ-functor $H_i(G,-)$ (Corollary 1.3), the following notion makes sense.

Definition 2.1 Let $H \to G$ be a group morphism. The *corestriction* morphisms

$$\mathrm{cor} = \mathrm{cor}_i^{H\to G} : H_i(H,M) \to H_i(G,M), \quad M \in {}_G\mathrm{Mod}$$

are the components of the unique homological δ-functor morphism $H_i(H,-) \to H_i(G,-)$ extending (2-1). ◆

So by contrast to cohomology, where *restriction* $\mathrm{res}^i_{G\to H} : H^i(G,-) \to H^i(H,-)$ was the "easy" concept to define (i.e. worked for any group morphism $H \to G$), in homology it is *co*restriction that is easier to come by.

2.2 Restriction

On the other hand, and again by analogy to cohomology, we also have "wrong way" morphisms

$$H_i(G,-) \to H_i(H,-) \tag{2-2}$$

provided $H \to G$ is a finite-index embedding. Assume that indeed this is the case for the duration of the present subsection §2.2.

The recipe for constructing (2-2) is parallel to that for defining corestriction in cohomology: we first start with $i = 0$, where we want maps

$$M_G \to M_H, \quad M \in {}_G\mathrm{Mod},$$

functorial in M.

Let t_i, $1 \le i \le d$ be a set of representatives for the *right* cosets $H\backslash G$. Then, for $m \in M$, define

$$\psi : M \ni m \mapsto \sum t_i m \in M. \tag{2-3}$$

We now come to

Problem 1 *Show that the class of the right hand side of (2-3) in M_H only depends on the class of m in M_G, and hence we have a commutative diagram*

$$\begin{tikzcd} & M \arrow[dr] & \\ M \arrow[ur, "\psi"] \arrow[dr] & & M_H \\ & M_G \arrow[ur, "\varphi"'] & \end{tikzcd}$$

for a unique map φ_M, functorial in $M \in {}_G\mathrm{Mod}$.

Now, by analogy to [1, Problem 3], one can prove that

$$H_i(H,-) : {}_G\mathrm{Mod} \to \mathrm{Ab},\ i \ge 0$$

form an effaceable (and hence universal) homological δ-functor (as in that problem, the subtlety here is that we're taking homology over H, but the modules are over G!). We can now make sense of

Definition 2.2 Let $H \le G$ be a finite-index subgroup. The *restriction* morphisms

$$\mathrm{res} = \mathrm{res}_i^{G\to H} : H_i(G,M) \to H_i(H,M), \quad M \in {}_G\mathrm{Mod}$$

are the components of the unique homological δ-functor morphism $H_i(G,-) \to H_i(H,-)$ extending the natural transformation $(-)_G \to (-)_H$ from Problem 1. ♦

We also have the following versions of [1, Theorem 2.4 and Corollary 2.5], with essentially the same proofs.

Theorem 2.3 *Let $H \le G$ be a finite-index subgroup. Then, for every G-module M, the composition*

$$\mathrm{cores} \circ \mathrm{res} : H_i(G,M) \to H_i(G,M)$$

is multiplication by the index $[G:H]$. ■

Corollary 2.4 *For any finite group G, the higher homology groups $H_i(G,-)$, $i \ge 1$ are annihilated by the order $|G|$.* ■

2.3 Transfer

Given a finite-index inclusion $H \le G$, we will now take a closer look at the restriction morphism

$$\mathrm{res}_1^{G\to H} : H_1(G) \to H_1(H).$$

It will take a particularly explicit form, since we know (from class meetings and, e.g., from [2, Theorem 9.5.2]) that the first homology group $H_1(G)$ is simply the abelianization

$$G_{ab} := G/[G,G].$$

So the general theory discussed above tells us that there should be an interesting morphism $G_{ab} \to H_{ab}$ arising as restriction in 1^{st} homology. This is the eponym of the present section:

Definition 2.5 For a finite-index subgroup $H \le G$, the *transfer* $V : G_{ab} \to H_{ab}$ is the restriction morphism

$$\mathrm{res}_1^{G\to H} : H_1(G) \cong G_{ab} \to H_{ab} \cong H_1(H).$$ ♦

Traditionally, the transfer is defined more explicitly (though it would then take some work to show that the construction is equivalent to Definition 2.5; see [2, Theorem 9.97]).

Let $d = [G:H]$ be the index and $\{s_i\}$ a set of representatives for the left cosets G/H. For every element $g \in G$ we have

$$gs_i = s_{\sigma i} h_i \tag{2-4}$$

for some permutation σ of $\{1,\cdots,n\}$ and some elements h_i. Then, set

$$V : G_{ab} \ni \text{ class of g} \mapsto \text{ class of } \prod_i h_i \in H_{ab}.$$

The problem with this is that

- it's not immediately clear it is a definition (i.e. only depends on the class of g in G_{ab} rather than on g itself);
- it's not clear it is independent of the set $\{s_i\}$;
- it's not clear it is a morphism.

So in other words, the only thing wrong with the construction is everything. Here, we'll walk through one possible way of addressing all of the issues that bypasses homology.

We write

$$\widetilde{V} : G \ni g \mapsto \text{ class of } \prod_i h_i \in H_{ab}.$$

(the domain is now G rather than G_{ab}). Note that $\widetilde{V}$ is at least well defined.

Definition 2.6 A *character* of H is a group morphism $\chi : H \to \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the multiplicative circle group of modulus-1 complex numbers.

Characters automatically factor through the abelianization $H \to H_{ab}$ (because their codomain $\mathbb{S}^1$ is abelian) and they form a group under pointwise multiplication.

The characters of H again form a group, with pointwise multiplication. We denote this group by

$$\widehat{H} := \text{ group of characters of } H,$$

and refer to it as the *Pontryagin dual* (or just 'dual', for short) group of H. ♦

A character χ gives an action of H on $\mathbb{C}$ (with $h \in H$ acting scaling $\mathbb{C}$ by $\chi(h)$), so you can regard $\mathbb{C}$ as an H-module denoted $\mathbb{C}_\chi$.

We can then extend scalars to get a G-module

$$M_\chi := \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_\chi$$

and we denote by

$$\rho_\chi : G \to Gl(M_\chi)$$

the resulting group morphism from G to the general linear group of the vector space M_χ. In other words, $\rho_\chi(g)$ is the operator g acting on the G-module M_χ.

Problem 2 *In the setup above, show that for every $\chi : H \to \mathbb{S}^1$ and every $g \in G$ we have*

$$\chi\left(\widetilde{V}(g)\right) = \sigma(g) \det \rho_\chi(g) \tag{2-5}$$

where $\sigma(g)$ is the sign of the permutation implemented by left multiplication by g on the set G/H of left cosets.

(2-5) is kind of cool in its own right (if you like that kind of thing), but we'll actually use it to address the three bullet points in the above discussion. Specifically, note that (2-5) implies that for every character $\chi \in \widehat{H_{ab}}$

- $\chi\left(\widetilde{V}(g)\right)$ only depends on the class of g in G_{ab} (why?);
- $\chi\left(\widetilde{V}(g)\right)$ does not depend on the choice of coset representatives s_i;

- the map
$$G \ni g \mapsto \chi\left(\widetilde{V}(g)\right) \in \mathbb{S}^1$$
is a morphism.

(these remarks precisely match the three bullet points listed before, in the same order).

Problem 3 *Conclude that the map $V : G \to H_{ab}$ defined by*
$$G \ni g \mapsto \text{ class of } \prod_i h_i \in H_{ab}$$
with h_i as in (2-4)

- *descends to a map $G_{ab} \to H_{ab}$;*
- *is independent of the coset representative set $\{s_i\}$;*
- *is a morphism $G_{ab} \to H_{ab}$.*

You already know from the discussion preceding the statement that all of these hold once you further compose V with arbitrary characters $\chi : H \to \mathbb{S}^1$. To conclude, you'll need to argue that characters *separate* the elements of H_{ab}: for every $x \neq y \in H_{ab}$ there is some character χ with $\chi(x) \neq \chi(y)$:

Problem 4 *Let H be an arbitrary abelian group. Show that the characters of H separate its elements in the sense of the preceding paragraph.*

Remark 2.7 A different way to phrase Problem 4 would have been: show that the canonical morphism $H \to \widehat{\widehat{H}}$ sending $h \in H$ to the morphism $\widehat{H} \to \mathbb{S}^1$ defined by
$$\chi \mapsto \chi(h)$$
is one-to-one. ♦

So I am suggesting you'll need Problem 4 to solve Problem 3. As for Problem 4 itself, all groups in sight are abelian now, so you're effectively working in the category Ab. I would recommend you use the fact that the circle group $\mathbb{S}^1$, where characters land by definition, is a divisible and hence injective abelian group (you can just use that result; it's a special case of [2, Corollary 3.35]).

I will wrap up with an illustration (or two..) of the usefulness of the transfer morphism. First, the following remark is a relatively simple consequence of the construction of $V \to G_{ab} \to H_{ab}$ in the discussion following Definition 2.5.

Proposition 2.8 *Let $H \leq G$ be a subgroup of finite index $d \in \mathbb{Z}_{\geq 1}$. Then, for each $g \in G$, the image of (the class of) g through the transfer morphism $V : G_{ab} \to H_{ab}$ is (the class in H_{ab} of) a product*
$$\prod_{i=1}^{s} x_i g^{n_i} x_i^{-1}$$
where

- *the exponents n_i add up to d;*

- *all* $x_i g^{n_i} x_i^{-1}$ *belong to* H.

■

That's a bit verbose, but there's a striking consequence that is much easier to state:

Corollary 2.9 *If* $H \le G$ *is a central subgroup of finite index* d *then the transfer morphism* $V : G_{ab} \to H$ *is simply*

$$G_{ab} \ni (\text{ class of } g \in G) \mapsto g^d \in H.$$

■

Note that it is not clear a priori, without the general theory of transfer morphisms, that the power-d map in Corollary 2.9 is even a morphism!

Finally, that application:

Proposition 2.10 *Let* G *be a finite group and* $H \le G$ *a finite central subgroup such that*

$$d := [G : H] \text{ and } m := |H|$$

are coprime. Then, G *decomposes as a direct product* $H \times N$.

Proof According to Corollary 2.9, the composition

$$\begin{tikzcd}
 & G_{ab} \arrow[dr, bend left, "V"] & \\
G \arrow[ur, bend left] \arrow[rr, bend right=10, "\widetilde{V}"'] & & H
\end{tikzcd}$$

of the transfer morphism with the canonical surjection $G \to G_{ab}$ is simply $x \mapsto x^d$. Because the order of the abelian group H is coprime to d, $\widetilde{V}$ restricts to an automorphism of H. It follows that for $N := \ker \widetilde{V}$ we indeed have the internal tensor product decomposition $G \cong H \times N$. ■

References

[1] Alexandru Chirvasitu. http://www.acsu.buffalo.edu/~achirvas/Math620_Spring2020/lec-2020-03-24.pdf.

[2] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.

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