# More about separation axioms

regular    completely regular    normal

$$T_3 < T_{3\frac{1}{2}} < T_4 \;\ldots\; - \ldots$$

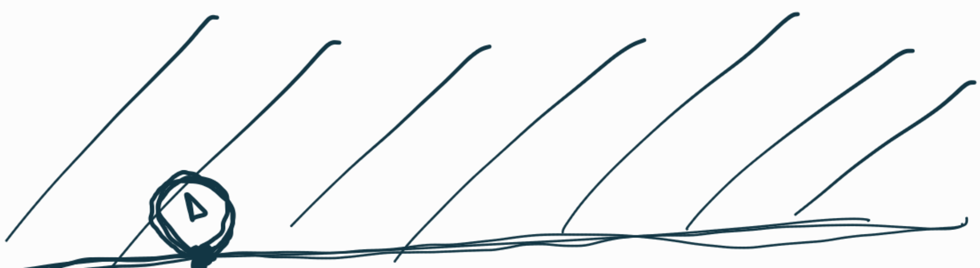
$T_3$ but not $T_4$ : Sorgenfrey plane $\mathbb{R}_\ell^2$

$T_{3\frac{1}{2}}$ not $T_4$

(search for 'completely regular' in Counterexamples in Topology by Steen-Seebach)

Niemytzki's tangent disk topology

As a set : $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$



$p$

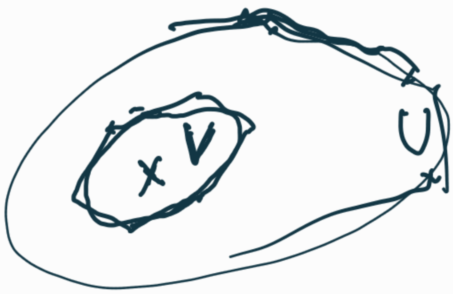
A basis of open sets consists of:
- open disks strictly in the upper half-plane
- sets of the form $p \cup \Delta$ where $p = (x, 0)$ for some $x \in \mathbb{R}$, $\Delta$ is an open disk in the upper half-plane, tangent to the bdry line at $p$

Alternative characterizations of regularity & normality

**Prop.** A $T_1$ top space $X$ is regular $\Longleftrightarrow$

(#) $\forall$ $x \in X$ & open $U \ni x$, $\exists$ open $V \ni x$ s.t. $\overline{V} \subseteq U$.

(#) Shrinking




Standard formulation of $T_3$:

$\forall\ x \in X$ , $\forall$ closed $F \not\ni x$

$\exists$ open sets $V \ni x$, $W \supseteq F$ disjoint

$V \subseteq W^c = \overline{W^c}$

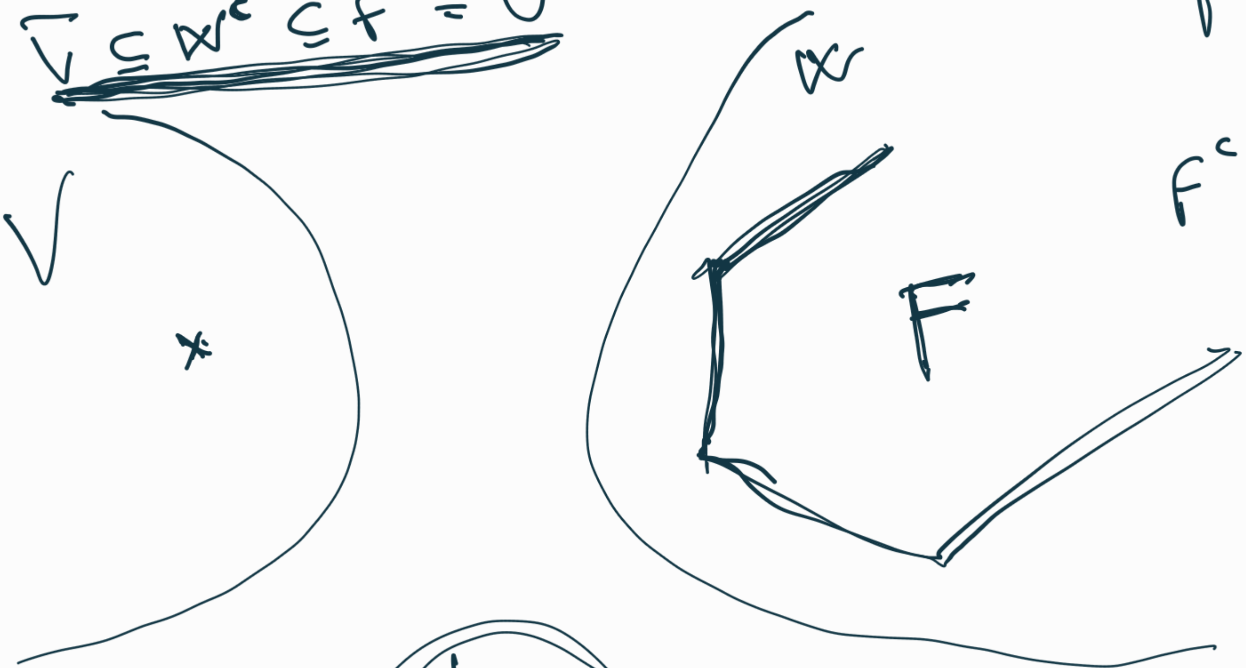
$\Downarrow$

$\bar{V} \subseteq W^c \subseteq F^c = U$

$F \subseteq W$

$\Downarrow$

$F^c \supseteq W^c$




$\Downarrow$

(#)

Let $x \in X$ & $\mathring{U} = U \ni x$

$W$ and $V$ w/ some properties

$U^c = X \setminus U$

Let $F = \ldots$

$F = \overline{F}$ b/c $U = U$

$F \not\ni x$ b/c $U \ni x$
$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\quad F^c$

$\Downarrow$ regularity

$\exists$ disjoint open $V \ni x$
$\qquad\qquad\qquad\quad W \supseteq F$

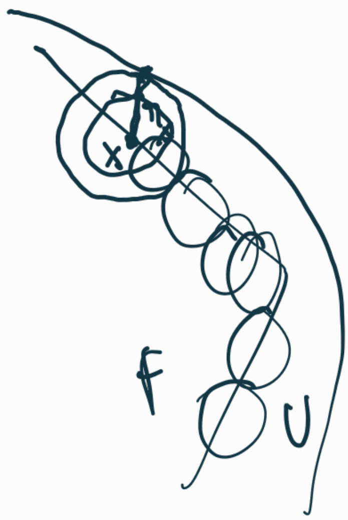
I claim that this $V$ does the job:

$V \ni x$ ✓
&
$\overline{V} \subseteq U$ ?

<u>Prop 2</u> A $T_1$ space $X$ is normal $\Longleftrightarrow$

$\forall$ closed $F \subseteq X$ & open $U \supseteq F$,

$\Rightarrow$ open $V \supseteq F$ s.t. $\overline{V} \subseteq U$


Metrizable spaces are normal: see lecture notes chapter 9

alternatively: use the characterization in

Prop 2: consider a closed set $F \subseteq (X, d)$

& an open $U \supseteq F$.

$x \in U$ $\quad\quad\quad\quad ) \subseteq U$

$\forall x \in F$, $\exists r_x > 0$ s.t. $B(x, r_x)$ ...

$\Downarrow U = \mathring{U}$

$\forall x \in F$, $\exists r_x > 0$ s.t.

Take $V = \bigcup_{x \in F} B(x, \frac{r_x}{2})$

Can be shown that $\overline{V} \subseteq U$.

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Countability axioms

"How few" elements (open sets, nbhds, whatever)

Can you ~~get~~ away with in defining your topology?

**Def** A set $S$ is countable if $\exists$ bijection $S \longrightarrow \mathbb{Z}$

$\mathbb{R}$ not countable

$2^{\mathbb{Z}}$ not countable

$$\mathbb{R} \cong 2^{\mathbb{Z}}$$

0 1 1 0 0 0 1 1 0 1 0

$$\frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \dots$$

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<u>Def</u>. A top. space $X$ is <u>first countable</u> (<u>separable</u>) if $X$ has a <u>dense</u> countable subset.

($Y \subseteq X$ is <u>dense</u> if $\overline{Y} = X$)

• $X$ is second countable if its topology has a countable basis.

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$(\mathbb{R}, \text{usual})$ is second countable:

$(a, b)$, $a < b \in \mathbb{Q}$ → countable

$$(\sqrt{2}, 3) = \bigcup_{\sqrt{2}+\varepsilon \in \mathbb{Q}} (\sqrt{2}+\varepsilon, 3)$$

Remark 2nd countable $\Rightarrow$ 1st countable:
$\not\Leftarrow$

If $\mathcal{B}$ is a countable basis for a top. $(X, \mathcal{T})$

pick a pt. $b \in B$ for each $B \in \mathcal{B}$

The collection of all such $\mathcal{B}$ is closed & countable

Ex $(\mathbb{R}, \text{Zariski})$ is - $1^{st}$ countable?

- not $2^{nd}$ countable

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