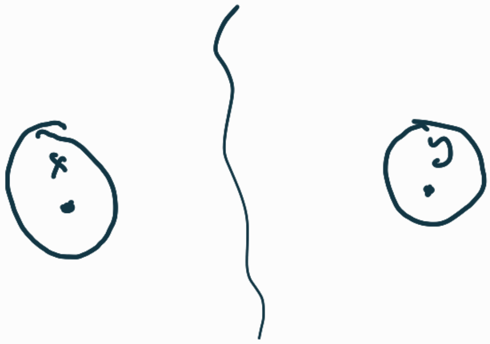
Separation axioms

How well can open sets "tell apart" pts or closed subsets of a top. space?

T_2 (the **Hausdorff** property): $\forall\ x \neq y \in X$

$$\begin{cases} \exists\, U, V \text{ open}, \\ x \in U \quad U \cap V = \emptyset \\ y \in V \end{cases}$$

Let $(X, \mathcal{T})$ be a top. space



X is T_1 if $\forall\ x \neq y \in X$, $\exists\, U, V$ open

$x \in U$

$y \in V$

$U \not\ni y$

$V \not\ni x$

Observation: $T_2 \Longrightarrow T_1$

$\overset{?}{\Longleftarrow}$

If '$\Leftarrow$' doesn't hold, should be able to give an example of a top. space that is T_1 but <u>not</u> T_2

<u>Ex</u> $\mathbb{R}$ w/ Zariski topology <u>is T_1</u> but <u>not T_2</u>

open sets: $\emptyset$, sets w/ finite complement
(<u>cofinite</u>)

Not T_2: If $U \ni x$, $V \ni y$ $\Rightarrow U, V \neq \emptyset \Rightarrow$ they are cofinite
$\Rightarrow$ they both contain the <u>cofinite</u> set $U \cap V$

T_1: $x \neq y$ $\quad$ $\mathbb{R} \setminus \{x\}$ contains y but not x
$\mathbb{R} \setminus \{y\}$ contains x but not y

X is T_0 (or Kolmogoroff) if

$\forall\ x \neq y \in X$, $\exists\ U$ open which contains exactly one of x & y

$$T_1 \Rightarrow T_0$$
$$\not\Leftarrow$$

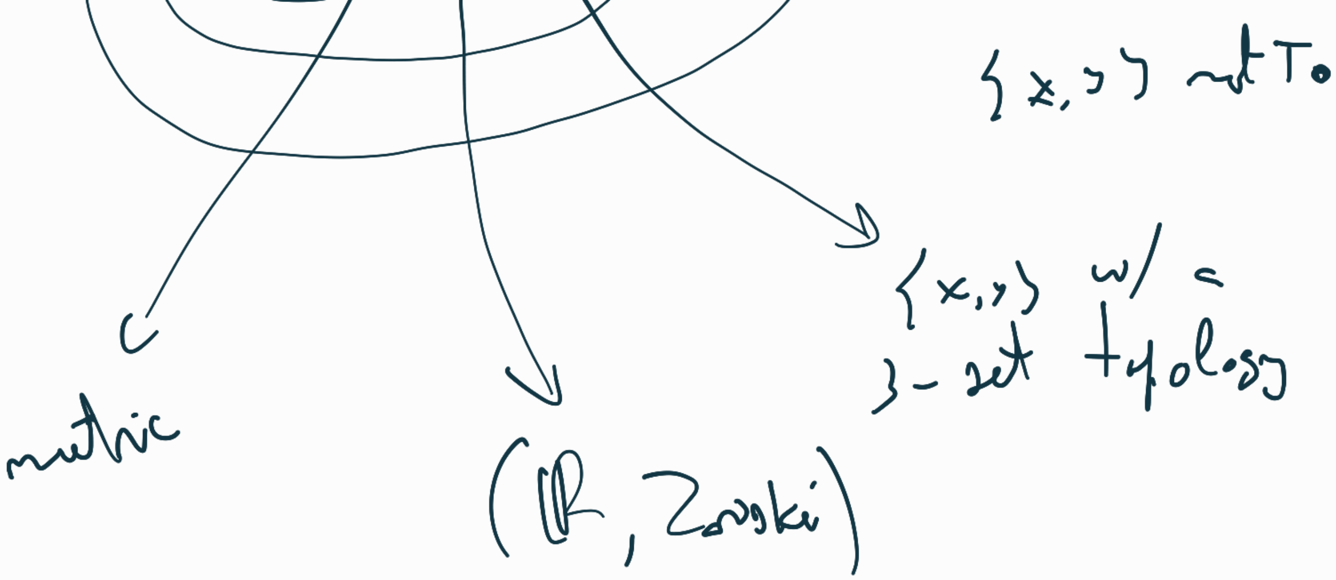
Ex T_0 space that is not T_1

$X = \{x, y\}$

Open sets: $\emptyset$, X, $\{x\}$

T_0: $U = \{x\}$ contains x but not y

not T_1: The only open set which contains y is $X \ni x$.



X is T_3 (or regular) if

(a) $\forall\ x \in X$, closed $F \subseteq X$ w/ $x \notin F$
$\exists$ open U, V s.t. $x \in U$, $F \subseteq V$, $U \cap V$ disjoint

(b) X is also T_1
(equivalently, singletons are closed)

Scratch: Indiscrete $X = \{x, y\}$ has property (a)
: the only closed sets are $\emptyset$ & $X \Rightarrow$

the only closed set not containing x is $\emptyset$ $\Rightarrow$ can take $U = X$, $V = \emptyset$

$T_3 \Rightarrow T_2$: apply (a) to x & $F = \{y\}$ $\quad (x \notin F \Leftrightarrow x \neq y)$

Check out Munkres's book (separation axioms)

<u>Ex</u> T_2 space, <u>not T_3</u>

$\mathbb{R}$ w/ basis consisting of
$$\begin{cases} (a,b), & a<b \\ (a,b) \setminus \underline{\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_{>0} \right\}} \end{cases}$$

On: Take the usual $\mathbb{R}$ & "make $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$ closed by fiat"

More general construction (Munkres): $\mathbb{R}_K$

$K \subseteq \mathbb{R}$ has as basis $\begin{cases} -(a,b) \\ -(a,b) \setminus K \end{cases}$

T_2: finer than ($\mathbb{R}$, usual) &

if $\mathcal{T}$ is finer than $\mathcal{T}'$ $\Rightarrow$ $\left\{ \Rightarrow \mathcal{T} \text{ is } T_2 \right.$

$\mathcal{T}'$ is T_2

Let $x \neq y \in (X, \mathcal{T}')$, $\mathcal{T}'$ is T_2

$\Rightarrow \exists\ U, V \in \mathcal{T}'$ $x \in U$
$y \in V$
$U \cap V = \emptyset$

let $\mathcal{T}$ is finer than $\mathcal{T}'$ $\Rightarrow$ $\mathcal{T} \supseteq \mathcal{T}'$

is open

so U & $V \in \tilde{\mathcal{J}} \Rightarrow \mathcal{J}$ is also T_2

not T_3: show that $\nexists$ disjoint open sets containing 0 & the closed set $\{\frac{1}{n} \mid n \in \mathbb{Z}_{\geq 0}\}$

$(X, \tilde{\mathcal{J}})$ is T_4 (or normal) if

(a) $\forall$ disjoint closed sets F & K, $\exists$ disjoint open $U \supseteq F$, $V \supseteq K$

(b) it is T_1 (or: singletons are closed; points are closed)

$T_4 \Rightarrow T_3$: take $K = \{x\}$

γ/

Ex T_3 space **and T_4**

$\mathbb{R}_\ell$ = Sorgenfrey line : base $[a,b)$, $a<b\in\mathbb{R}$
(finer than standard)

is T_4

$\mathbb{R}^2_\ell = \mathbb{R}_\ell \times \mathbb{R}_\ell$ (Sorgenfrey plane)

T_3 but not T_4

$\mathbb{R}^2_\ell$ has **basis** $[a,b) \times [c,d)$

Remark

We noticed that the T_3 property is preserved by passage to a finer topology

T_3 or T_4 !

Not true of T_3

- $\mathbb{R}_{\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}}$ is finer than $(\mathbb{R}, \text{usual})$

  $\mathbb{R}_{\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}}$ $\rightsquigarrow$ not T_3

  $(\mathbb{R}, \text{usual})$ $\rightsquigarrow$ T_3

- $\mathbb{R}^2_\ell$ is finer than $(\mathbb{R}^2, \text{usual})$

  $\mathbb{R}^2_\ell$ $\rightsquigarrow$ not T_4

  $(\mathbb{R}^2, \text{usual})$ $\rightsquigarrow$ T_4 b/c metrizable

  metric spaces are T_4

Sometimes want to construct continuous fns on them taking pre-specified values along closed sets. T_4 spaces are best for this:

Urysohn's Lemma

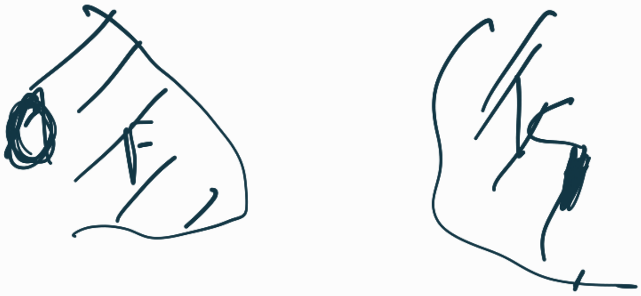
Let $(X, \mathcal{T})$ be a T_4 space &

$\underbrace{F \cap K}_{\text{closed}} = \emptyset$

$\exists$ cont. function $f : X \longrightarrow [0,1]$

s.t. $f\big|_F \equiv 0$

$f\big|_K \equiv 1$



$T_{3\,1/2}$ (complete regularity):

$\forall\, x$, $\forall$ closed $F \not\ni x$, $\exists$ cont. $f : X \longrightarrow (0,1)$

$f(x) = 0$

$f\big|_F \equiv 1$