6 | Continuous Functions

Let X, Y be topological spaces. Recall that a function $f: X \to Y$ is continuous if for every open set $U \subseteq Y$ the set $f^{-1}(U) \subseteq X$ is open. In this chapter we study some properties of continuous functions. We also introduce the notion of a *homeomorphism* that plays a central role in topology: from the topological perspective interesting properties of spaces are the properties that are preserved by homeomorphisms.

6.1 Proposition. *Let X, Y be topological spaces. A function $f: X \to Y$ is continuous if and only if for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed.*

Proof. Assume that $f: X \to Y$ is a continuous function and let $A \subseteq Y$ be a closed set. We have

$$f^{-1}(A) = X \smallsetminus f^{-1}(Y \smallsetminus A)$$

The set $Y \smallsetminus A$ is open in Y so by continuity of f the set $f^{-1}(Y \smallsetminus A) \subseteq X$ is open in X. It follows that $f^{-1}(A)$ is closed in X.

Conversely, assume that $f: X \to Y$ is a function such that for every closed set $A \subseteq Y$ the set $f^{-1}(A) \subseteq X$ is closed. Let $U \subseteq Y$ be an open set. We have

$$f^{-1}(U) = X \smallsetminus f^{-1}(Y \smallsetminus U)$$

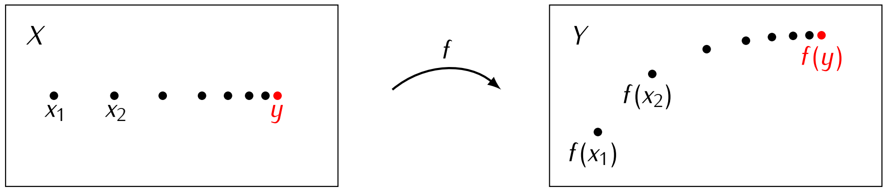
The set $Y \smallsetminus U$ is closed in Y so by assumption the set $f^{-1}(Y \smallsetminus U)$ is closed in X. If follows that $f^{-1}(U)$ is open in X. Therefore f is a continuous function. □

For metric spaces continuous functions are precisely the functions that preserve convergence of sequences:

6.2 Proposition. *Let (X, ϱ) be a metric space, let Y be a topological space, and let $f: X \to Y$ be a function. The following conditions are equivalent:*

1) f is continuous.

2) For any sequence $\{x_n\} \subseteq X$ *if* $x_n \to y$ *for some* $y \in X$ *then* $f(x_n) \to f(y)$.



Proof. 1) $\Rightarrow$ 2) Exercise.

2) $\Rightarrow$ 1) Let $A \subseteq Y$ be a closed set. We will show that the set $f^{-1}(A)$ is closed in X. By Proposition 5.8 it suffices to show that if $\{x_n\} \subseteq f^{-1}(A)$ is a sequence and $x_n \to x$ then $x \in f^{-1}(A)$.

If $x_n \to x$ then by assumption we have $f(x_n) \to f(x)$. Since $\{f(x_n)\} \subseteq A$ and A is a closed set, thus by Proposition 5.8 we obtain that $f(x) \in A$, and so $x \in f^{-1}(A)$. □

The implication 1) $\Rightarrow$ 2) in Proposition 6.2 holds for maps between general topological spaces:

6.3 Proposition. *Let* $f\colon X \to Y$ *be a continuous function of topological spaces. If* $\{x_n\} \subseteq X$ *is a sequence and* $x_n \to x$ *for some* $x \in X$ *then* $f(x_n) \to f(x)$.

Proof. Exercise. □

6.4 Example. We will show that the implication 2) $\Rightarrow$ 1) in Proposition 6.2 is not true if X is a general topological space. Let X be the space defined in Example 5.16: $X = \mathbb{R}$ with the topology

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U = \varnothing \text{ or } U = (\mathbb{R} \smallsetminus S) \text{ for some countable set } S \subseteq \mathbb{R}\}$$

Recall that if $\{x_n\}$ is a sequence in X then $x_n \to x$ if and only if there exists $N > 0$ such that $x_n = x$ for all $n > N$. Let $f\colon X \to X$ be a function given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) \\ 1 & \text{if } x \notin (0,1) \end{cases}$$

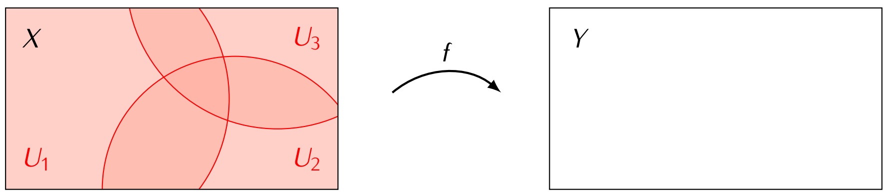
This function is not continuous since the set $\{0\}$ is closed in X and the set $(0,1) = f^{-1}(\{0\})$ is not closed in X. On the other hand let $\{x_n\} \subseteq X$ be a sequence and let $x_n \to x$. There is $N > 0$ such that $x_n = x$ for $n > N$, so $f(x_n) = f(x)$ for all $n > N$ and so $f(x_n) \to f(x)$.

6.5 Proposition. *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are continuous functions then the function* $gf\colon X \to Z$ *is also continuous.*

Proof. Exercise. □

Frequently functions $f\colon X \to Y$ are constructed by gluing together several functions defined on subspaces of X. The next two facts are useful for verifying that functions obtained in this way are continuous.

6.6 Open Pasting Lemma. *Let X, Y be topological spaces and let $\{U_i\}_{i\in I}$ be a family of open sets in X such that $\bigcup_{i\in I} U_i = X$. Assume that for $i \in I$ we have a continuous function $f_i\colon U_i \to Y$ such that $f_i(x) = f_j(x)$ if $x \in U_i \cap U_j$. Then the function $f\colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in U_i$ is continuous.*



Proof. Let $V \subseteq Y$ be an open set. We will show that the set $f^{-1}(V) \subseteq X$ is open. Since $\bigcup_{i\in I} U_i = X$ we have

$$f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V) \cap U_i = \bigcup_{i\in I} f_i^{-1}(V)$$

Since $f_i\colon U_i \to Y$ is a continuous function the set $f_i^{-1}(V)$ is open in U_i. Also, since U_i is open in X by Exercise 5.8 we obtain that the set $f_i^{-1}(V)$ is open in X. Thus $f^{-1}(V)$ is an open set. □

6.7 Closed Pasting Lemma. *Let X, Y be topological spaces and let $A_1, \dots, A_n \subseteq X$ be a finite family of closed sets such that $\bigcup_{i=1}^{n} A_i = X$. Assume that for $i = 1, 2, \dots, n$ we have a continuous function $f_i\colon A_i \to Y$ such that $f_i(x) = f_j(x)$ if $x \in A_i \cap A_j$. Then the function $f\colon X \to Y$ given by $f(x) = f_i(x)$ for $x \in A_i$ is continuous.*

Proof. Exercise. □

6.8 Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the absolute value function, $f(x) = |x|$. On the set $A_1 = (-\infty, 0]$ this function is given by $f|_{A_1}(x) = -x$, and on $A_2 = [0, +\infty)$ it is given by $f|_{A_2}(x) = x$. Since both $f|_{A_1}$ and $f|_{A_2}$ are continuous functions and A_1, A_2 are closed sets in $\mathbb{R}$ by the Closed Pasting Lemma 6.7 we obtain that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

6.9 Definition. A *homeomorphism* is a continuous function $f\colon X \to Y$ such that f is a bijection and the inverse function $f^{-1}\colon Y \to X$ is continuous.

6.10 Proposition. *1) For any topological space the identify function $\mathrm{id}_X\colon X \to X$ given by $\mathrm{id}_X(x) = x$ is a homeomorphism.*

2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms then the function $gf\colon X \to Z$ is also a homeomorphism.

3) If $f\colon X \to Y$ is a homeomorphism then the inverse function $f^{-1}\colon Y \to X$ is also a homeomorphism.

4) If $f\colon X \to Y$ is a homeomorphism and $Z \subseteq X$ then the function $f|_Z\colon Z \to f(Z)$ is also a homeomorphism.

Proof. Exercise. □

6.11 Note. If $f\colon X \to Y$ is a continuous bijection then f need not be a homeomorphism since the inverse function f^{-1} may be not continuous. For example, let $X = \{x_1, x_2\}$ be a space with the discrete topology and let $Y = \{y_1, y_2\}$ be a space with the antidiscrete topology. Let $f\colon X \to Y$ be given by $f(x_i) = y_i$. The function f is continuous but f^{-1} is not continuous since the set $\{x_1\}$ is open in X, but the set $(f^{-1})^{-1}(\{x_1\}) = \{y_1\}$ is not open in Y.

6.12 Proposition. *Let $f\colon X \to Y$ be a continuous bijection. The following conditions are equivalent:*

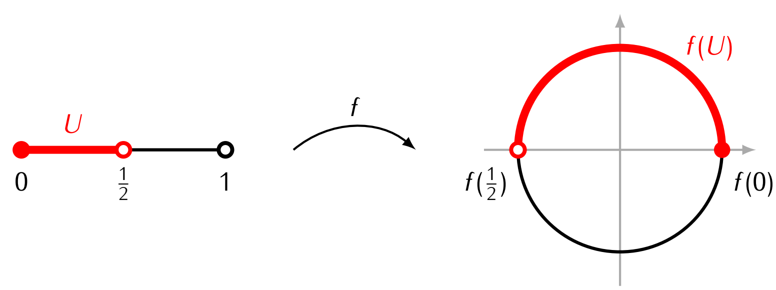
(i) The function f is a homeomorphism.
(ii) For each open set $U \subseteq X$ the set $f(U) \subseteq Y$ is open.
(iii) For each closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed.

Proof. Exercise. □

6.13 Example. Recall that S^1 denotes the unit circle:

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

The function $f\colon [0,1) \to S^1$ given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a continuous bijection, but it is not a homeomorphism since the set $U = [0, \frac{1}{2})$ is open in $[0,1)$, but $f(U)$ is not open in S^1.

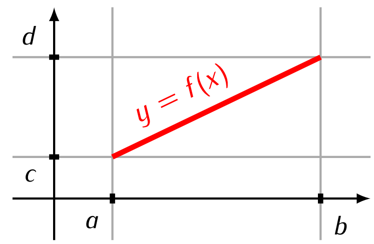


6.14 Definition. We say that topological spaces X, Y are *homeomorphic* if there exists a homeomorphism $f\colon X \to Y$. In such case we write: $X \cong Y$.

6.15 Note. Notice that if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

6.16 Example. For any $a < b$ and $c < d$ the open intervals $(a, b), (c, d) \subseteq \mathbb{R}$ are homeomorphic. To see this take e.g. the function $f\colon (a, b) \to (c, d)$ defined by

$$f(x) = \left(\frac{c-d}{a-b}\right) x + \left(\frac{ad-bc}{a-b}\right)$$



This function is a continuous bijection. Its inverse function $f^{-1}\colon (c, d) \to (a, b)$ is given by

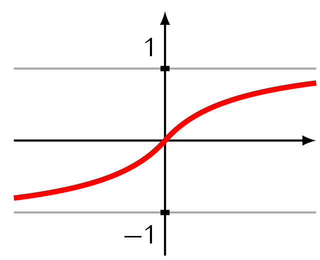
$$f^{-1}(x) = \left(\frac{a-b}{c-d}\right) x + \left(\frac{cb-da}{c-d}\right)$$

so it is also continuous. By the same argument for any $a < b$ and $c < d$ the closed intervals $[a, b], [c, d] \subseteq \mathbb{R}$ are homeomorphic.

6.17 Note. In Chapter 7 we will show that an open interval (a, b) is not homeomorphic to a closed interval $[c, d]$.

6.18 Example. We will show that for any $a < b$ the open interval (a, b) is homeomorphic to $\mathbb{R}$. Since $(a, b) \cong (-1, 1)$ it will be enough to check that $\mathbb{R} \cong (-1, 1)$. Take the function $f\colon \mathbb{R} \to (-1, 1)$ given by

$$f(x) = \frac{x}{1+|x|}$$



This function is a continuous bijection with the inverse function $f^{-1}\colon (-1, 1) \to \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x}{1-|x|}$$

Since f^{-1} is continuous we obtain that f is a homeomorphism.

6.19 Note. If spaces X and Y are homeomorphic then usually there are many homeomorphisms $X \to Y$. For example, the function $g\colon (-1, 1) \to \mathbb{R}$ given by

$$g(x) = \tan\left(\tfrac{\pi}{2}x\right)$$

is another homeomorphism between the spaces $(-1, 1)$ and $\mathbb{R}$.

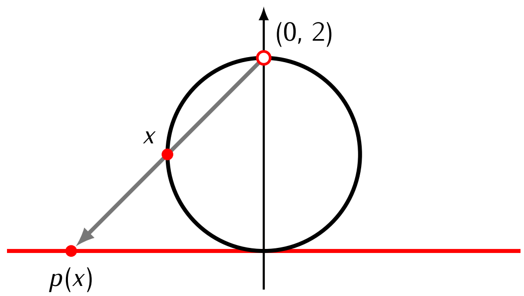
6.20 Example. We will show that for any point $x_0 \in S^1$ there is a homeomorphism $S^1 \smallsetminus \{x_0\} \cong \mathbb{R}$. Denote by $S^1_{(0,1)} \subseteq \mathbb{R}$ the circle of radius 1 with the center at the point $(0, 1) \in \mathbb{R}^2$:

$$S^1_{(0,1)} := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 = 1\}$$

It is easy to check that for $x_0 \in S^1$ the space $S^1 \smallsetminus \{x_0\}$ is homeomorphic to the space $X = S^1_{(0,1)} \smallsetminus \{(0, 2)\}$. Likewise, it is easy to check that $\mathbb{R}$ is homeomorphic to the subspace $Y \subseteq \mathbb{R}^2$ that consists of all points of the x-axis:

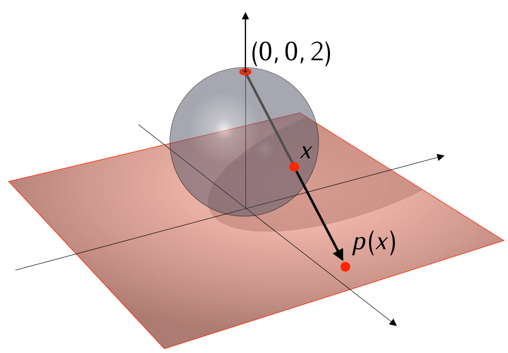
$$Y := \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}$$

It is then enough to show that $X \cong Y$. A homeomorphism $p\colon X \to Y$ can be constructed as follows. For any point $x \in X$ there is a unique line in $\mathbb{R}^2$ that passes through x and though the point $(0, 2) \in \mathbb{R}^2$. We define $p(x)$ to be the point of intersection of this line with the x-axis:



The function p is called the *stereographic projection*.

In a similar way we can construct a stereographic projection in any dimension $n \geq 1$ that gives a homeomorphism between the space $S^n \smallsetminus \{x_0\}$ (i.e. the n-dimensional sphere with one point deleted) and the space $\mathbb{R}^n$:



Exercises to Chapter 6

E6.1 Exercise. Consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $\mathbb{Q}$ is not homeomorphic to a space with the discrete topology.

E6.2 Exercise. Prove Proposition 6.3.

E6.3 Exercise. Prove Proposition 6.5.

E6.4 Exercise. Prove Lemma 6.7.

E6.5 Exercise. Prove Proposition 6.12.

E6.6 Exercise. Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions.

a) Show that the set
$$A = \{x \in X \mid f(x) \geq g(x)\}$$
is closed in X.

b) Let $h_{\mathrm{max}}, h_{\mathrm{min}}\colon X \to \mathbb{R}$ be a functions given by $h_{\mathrm{max}}(x) = \max\{f(x), g(x)\}$ and $h_{\mathrm{min}}(x) = \min\{f(x), g(x)\}$. Show that h_{max} and h_{min} are continuous functions.

E6.7 Exercise. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) > g(x)$ for all $x \in \mathbb{R}$. Define subspaces X, Y of $\mathbb{R}^2$ as follows.
$$X := \{(x, y) \in \mathbb{R}^2 \mid g(x) \leq y \leq f(x)\} \qquad Y := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$$
Show that $X \cong Y$.

E6.8 Exercise. Let $x_0 = (0, 0) \in \mathbb{R}^2$ and let $\overline{B}(x_0, 1) \subseteq \mathbb{R}^2$ be a closed ball defined by the Euclidean metric d:
$$\overline{B}(x_0, 1) = \{x \in \mathbb{R}^2 \mid d(x, x_0) \leq 1\}$$
Define subspaces $X, Y \subseteq \mathbb{R}^2$ as follows:
$$X := \mathbb{R}^2 \smallsetminus \{x_0\} \qquad Y := \mathbb{R}^2 \smallsetminus \overline{B}(x_0, 1)$$
Show that $X \cong Y$.

E6.9 Exercise. Let (X, ϱ) be a metric space. A subspace $Y \subseteq X$ is a *retract* of X if there exists a continuous function $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$. Show that if $Y \subseteq X$ is a retract of X then Y is a closed in X.