Math 424 B / 574 B Wednesday, Oct 28

Autumn 2015

Midterm

Name: ____________________

Student ID: ____________________

1	15	
2	15	
Total	30+5=35	

- **Please read this.**
- You have 50 minutes.
- This is an open book exam. You can't collaborate (with anyone, either in class or electronically), but you're allowed any reading materials you like.
- Show all of your work and justify your answers unless instructed not to do so.
- If you want to use a theorem from the book, cite it by number; if you're using a homework problem, tell me which one.
- Not citing a previous result you're using could lose you points.
- Use the reverse side of each page for extra space and add paper of your own if you run out.

Problem 1. *Let (X, d) be a metric space, $E \subseteq X$ a subset. As usual, we denote the interior, closure and complement of E by E°, $\overline{E}$ and E^c respectively.*

Define the set ∂E as follows:

$$\partial E = \overline{E} \cap \overline{E^c}.$$

(a) *Show that $\partial E = \overline{E} \setminus E^\circ$.*
(b) *Show that the three sets E°, ∂E and $(E^c)^\circ$ are mutually disjoint (meaning no two of them intersect).*
(c) *Show that we have*

$$X = E^\circ \cup \partial E \cup (E^c)^\circ.$$

Solution.

(a) From the definition of ∂E you know that it is contained in $\overline{E}$. To prove (a) we have to prove that for any $x \in \overline{E}$ we have

$$x \in \overline{E^c} \Leftrightarrow x \notin E^\circ. \tag{1}$$

Indeed, $x \in \overline{E}$ satisfies the left hand side of (1) if and only if it belongs to

$$\overline{E} \cap \overline{E^c} = \partial E,$$

and it satisfies the right hand side if and only if it belongs to $\overline{E} \setminus E^\circ$ by the definition of set difference.

In fact, (1) is true for any $x \in X$ whatsoever (not just for $x \in \overline{E}$). This is a reformulation of part (d) from Problem 9, page 43 of our textbook (which was assigned as homework).

(b) First, note that E and E^c play the same role in the definition of ∂E, so that $\partial E = \partial(E^c)$.

Now, part (a) implies that ∂E is disjoint from E°. On the other hand, interchanging the roles of E and E^c, we also know that $\partial E = \partial(E^c)$ is disjoint from $(E^c)^\circ$. Finally, E° and $(E^c)^\circ$ are disjoint because they are contained in the disjoint sets E and E^c respectively.

(c) Applying part (a) to E^c instead of E and using the observation made before that $\partial E = \partial(E^c)$, we get

$$\overline{E^c} = \partial(E^c) \cup (E^c)^\circ = \partial E \cup (E^c)^\circ. \tag{2}$$

On the other hand, part (d) of Problem 9 on page 43, which I've cited before a few paragraphs back, says that

$$\overline{E^c} = (E^\circ)^c,$$

and so

$$X = E^\circ \cup \overline{E^c}. \tag{3}$$

Plugging (2) into the right hand side of (3) gives the desired result. ■

Problem 2. *Let (X, d) be a metric space, and $E \subseteq X$ a subset.*

E is said to be **constrained** *if for every $\delta > 0$ there exist finitely many points $x_1, \cdots, x_n \in X$ such that*

$$E \subseteq \bigcup_{i=1}^{n} N_\delta(x_i).$$

(a) *Show that every compact subset of X is constrained.*
(b) *Show that every constrained subset of X is bounded.*
(c) *Show that every bounded subset of $\mathbb{R}$ is constrained.*
(d) **(Bonus: 2 extra points on top of the 35, but no partial credit)** *Show by example that a bounded subset of a metric space is not necessarily constrained.*

Remark 3. Before going into the proof, a word on terminology: the "official" term for 'constrained' that is used in the mathematical literature is **totally bounded**. I made up another one to make it harder for you to cheat by using the internet, which you had access to during the test (proofs of pretty much all of the statements can be found online).

In the solution I will switch freely between 'constrained' and 'totally bounded'. ♦

Solution.

(a) Let $K \subseteq X$ be a compact set and fix an arbitrary $\delta > 0$. The neighborhoods $N_\delta(x)$ of radius δ centered at points $x \in K$ form an open cover of K (because each $x \in K$ is covered by its own neighborhood $N_\delta(x)$).

By compactness, I can extract a finite subcover

$$K \subseteq \bigcup_{i=1}^{n} N_\delta(x_i)$$

for $x_i \in K$. Since this works for any $\delta > 0$, the definition of being constrained (or totally bounded) is satisfied by K.

(b) Let $A \subseteq X$ be a totally bounded subset. According to the definition of total boundedness in the statement of the problem, I can find finitely many points $x_1, \cdots, x_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} N_1(x_i)$$

(there is nothing special about the radius 1; any positive δ will do for this argument, but I wanted to make a concrete choice).

Now, for every $a \in A$ we have $a \in N_1(x_j)$ for some $1 \le j \le n$. Using the triangle inequality, we then have

$$d(a, x_1) \le d(a, x_j) + d(x_j, x_1) < 1 + d(x_j, x_1) \le 1 + \max_{i=1}^{n} d(x_i, x_1).$$

Denoting this last expression by R, we have

$$d(a, x_1) < R, \ \forall a \in A,$$

and hence A is contained in the neighborhood of radius R centered at x_1. According to the definition of boundedness, this means that A is indeed bounded.

(c) A bounded subset A of $\mathbb{R}$ is contained in some interval $[a, b]$, for real numbers $a < b$. We know from Theorem 2.40 in the textbook that $[a, b]$ is compact, and hence totally bounded by part (a). Finally, subsets of totally bounded sets are totally bounded (this is immediate from the definition), so A is totally bounded.

(d) We talked in class about something called the **discrete metric** on a set X: simply define $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$. It is easy to check that this is a metric.

Now take (X, d) to be an infinite set equipped with the discrete metric. It is bounded because for every $x \in X$ we have $X = N_2(x)$ (since for every $y \in x$ we have $d(x, y) \leq 1 < 2$). On the other hand, I cannot cover X with finitely many neighborhoods of radius 1. Indeed, for every $x \in X$ we have $N_1(x) = \{x\}$, so that we need infinitely many $N_1(x)$ to cover the (infinitely many) points of X.

So choosing $\delta = 1$ violates the definition of being constrained, and hence X cannot be constrained (when regarded as a subset of (X, d)). $\blacksquare$