Math 424 B / 574 B Due Wednesday, Dec 09
Autumn 2015

Selected solutions for Homework 9

This file includes solutions only to those problems we did not have time to cover in class.

We need the following concept.

Definition 1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is *bounded* if $f(X)$ is a bounded subset of Y in the sense of Definition 2.18 (i) in our textbook. ⧫

E1. Let (X, d_X) be a metric space, and let $\mathcal{B}(X)$ be the set of all functions from X to $\mathbb{R}$ that are bounded in the sense of Definition 1. For $f, g \in \mathcal{B}(X)$ define

$$d(f, g) = \sup\{|f(x) - g(x)| : \ x \in X\}.$$

(a) Show that $d(f, g)$ is a non-negative number for all $f, g \in \mathcal{B}(X)$ (that is, it's never ∞).
(b) Show that $\mathcal{B}(X)$ equipped with d is a metric space.
(c) Prove that the subset $\mathcal{CB}(X) \subset \mathcal{B}(X)$ consisting of those bounded functions $X \to \mathbb{R}$ which are also continuous is closed in the metric space $(\mathcal{B}(X), d)$ from part (b).
(d) Show that the metric space $(\mathcal{B}(X), d)$ is complete.

Solution. We did parts (a) and (b) in class, and got started on part (d). I will only fill in the missing pieces.

(c) The goal here is to prove that if $f_n \in \mathcal{B}(X)$ is a sequence of continuous bounded functions and $f_n \to f$ in the metric space $\mathcal{B}(X)$, then the function f is also continuous. To do this, I will use the sequential characterization of continuity: a function $g : X \to \mathbb{R}$ is continuous if and only if

$$x_m \to x \text{ in } X \Rightarrow g(x_m) \to g(x) \text{ in } \mathbb{R}.$$

Start with some sequence $(x_m)_m$ in X which converges to a point $x \in X$. According to the above characterization of continuity, I want to prove that $f(x_m)$ converges to $f(x)$ in $\mathbb{R}$, meaning that for any $\varepsilon > 0$ there is some positive integer N such that

$$m \geq N \Rightarrow |f(x_m) - f(x)| \leq \varepsilon. \tag{1}$$

Now, I know that $f_n \to f$ in $(\mathcal{B}(X), d)$, which means that I can find some n such that $d(f_n, f) \leq \frac{\varepsilon}{3}$. According to the definition of the metric d on $\mathcal{B}(X)$, this the same as saying that

$$|f_n(p) - f(p)| \leq \frac{\varepsilon}{3}, \ \forall p \in X. \tag{2}$$

Furthermore, because f_n is continuous and $x_m \to x$, there is some positive integer N such that

$$m \geq N \Rightarrow |f_n(x_m) - f_n(x)| \leq \frac{\varepsilon}{3}. \tag{3}$$

Now, using the triangle inequality, we get

$$|f(x_m) - f(x)| \leq |f(x_m) - f_n(x_m)| + |f_n(x_m) - f_n(x)| + |f_n(x) - f(x)|.$$

When $m \geq N$ the term in the middle of the right hand sum is $\leq \frac{\varepsilon}{3}$ by (3). On the other hand, the two outer terms in this three-term sum are both $\leq \frac{\varepsilon}{3}$ by applying (2) to $p = x_m$ and $p = x$ respectively. So all in all, we get the implication (1), which is what we were after.

(d) I have to show that every Cauchy sequence $(f_n)_n \subset (\mathcal{B}(X), d)$ is convergent to some $f \in \mathcal{B}(X)$. Here's where we left off in class: we observed that for every $x \in X$ the sequence $(f_n(x))_n$ of real numbers is Cauchy, and hence convergent because $\mathbb{R}$ is complete (Theorem 3.11). Then we *defined* $f(x)$ to be the limit of $(f_n(x))_n$.

So we now have a function $f : X \to \mathbb{R}$, and we need to prove two things about it:

First, that it is an element of $\mathcal{B}(X)$ (in other words, it is bounded). Secondly, that we do indeed have $f_n \to f$ in $\mathcal{B}(X)$ with respect to the metric d (We don't know that yet! We just know that each $f_n(x)$ converges to $f(x)$).

Proof that f is bounded. By the Cauchy property for the sequence (f_n), there is some positive integer N such that $d(f_n, f_m) \leq 1$ for all $n, m \geq N$. But by the definition of the distance d on $\mathcal{B}(X)$, this implies

$$|f_n(x) - f_N(x)| \leq 1, \ \forall n \geq N, \ \forall x \in X.$$

So for all points $x \in X$, the terms $f_n(x)$, $n \geq N$ are contained in the interval

$$I_x = [f_N(x) - 1, f_N(x) + 1].$$

This implies that the limit $f(x)$ of $(f_n(x))_n$ is also contained in I_x for every x, and hence

$$|f(x) - f_N(x)| \leq 1, \ \forall x \in X. \tag{4}$$

Now remember furthermore that f_N was itself bounded, so $|f_N(x)| \leq M$, $\forall x$ for some positive real M. This, together with (4) gives

$$\forall x \in X, \ |f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M.$$

This finishes the proof that f is indeed bounded.

Proof that $f_n \to f$ in $\mathcal{B}(X)$. Fix $\varepsilon > 0$; we have to come up with some positive integer N so that

$$n \geq N \Rightarrow d(f_n, f) \leq \varepsilon$$

which in turn, from the definition of d, is the same as proving that

$$n \geq N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon \text{ for all } x \in X. \tag{5}$$

The proof is very similar to the one above, for the boundedness of f. By The Cauchy property for the sequence (f_n), there is some positive integer N such that $d(f_n, f_m) \leq \varepsilon$ for all $n, m \geq N$. In other words,

$$|f_n(x) - f_m(x)| \leq \varepsilon \Rightarrow f_m(x) \in [f_n(x) - \varepsilon, f_n(x) + \varepsilon]$$

for all $x \in X$ and all $n, m \geq N$. But then, for every $x \in X$, the limit $f(x)$ of the sequence $(f_m(x))_m$ is also contained in the closed interval $[f_n(x) - \varepsilon, f_n(x) + \varepsilon]$ and we get the desired inequality (5). ∎

A word about the proof of part (d). It was slightly subtle that even though we knew that $f_n(x) \to f(x)$ for each individual x, we still had to prove that $f_n \to f$ with respect to the metric d on $\mathcal{B}(X)$. These are two different types of convergence that are not equivalent to one another. Here's the relevant definition:

Definition 2. Let $f_n, f : X \to Y$ be functions between two metric spaces (X, d_X) and (Y, d_Y). We say that $(f_n)_n$ *converges pointwise* to f if $f_n(x) \to f(x)$ in Y for each individual point $x \in X$.

When f_n and f are bounded in the sense of Definition 1 we say that $(f_n)_n$ *converges uniformly* to f if $f_n \to f$ with respect to the metric d defined by

$$d(g, h) = \sup\{d_Y(g(x), h(x)) : \ x \in X\},$$

analogous to the distance d from the statement of E1. ♦

The subtlety in the proof of part (d) above stems from the fact that we initially had pointwise convergence, but that doesn't, on its own, imply uniform convergence (which is what we wanted). The following example illustrates this.

Example 3. Take say $X = (0, \infty)$ and $Y = \mathbb{R}$ with the usual metrics, so we are looking at real-valued functions defined on $(0, \infty)$. For each positive integer n define

$$f_n(x) = \begin{cases} 1 & \text{if } x \le \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{cases}$$

and define f to be identically zero. Then $f_n \to f$ pointwise, because every $x \in (0, \infty)$ will be caught inside the intervals $\left(\frac{1}{n}, \infty\right)$ for large n, and for those n the values $f_n(x)$ stabilize to $0 = f(x)$. On the other hand though we have

$$\left| f_n\left(\frac{1}{2n}\right) - f\left(\frac{1}{2n}\right) \right| = |1 - 0| = 1,$$

so $d(f_n, f) \ge 1$ for all n. In particular, f_n *does not* converge to f uniformly. ♦

It can be shown, however, that uniform boundedness always implies pointwise boundedness. So: uniform is *strictly stronger* than pointwise.

E3. Is there a continuous function $f : (-2, 6) \cup (6, 10) \to \mathbb{R}$ for which $f(6-)$ and $f(6+)$ both exist and such that $f(6-) < f(6+)$? How about a *uniformly* continuous one?

Solution. There are two questions: the answer to the first one is 'yes' (there is a continuous function with that property), while the answer to the second one is 'no' (you can't make such a function *uniformly* continuous).

I will denote the domain $(-2, 6) \cup (6, 10)$ of our functions by D.

Continuous f. Just take f to be, say constantly 0 on $(-2, 6)$ and constantly 1 on $(6, 10)$. Then you have

$$f(6-) = 0 < 1 = f(6+),$$

and the function is continuous because for every point in its domain D its restriction to some small neighborhood of that point is constant (and hence continuous).

Uniformly continuous f. The goal is to show that if $f : D \to \mathbb{R}$ is uniformly continuous and $f(6-)$ and $f(6+)$ both exist, then $f(6-) = f(6+)$.

Fix some $\varepsilon > 0$ we'll show that

$$|f(6-) - f(6+)| < \varepsilon; \tag{6}$$

since $\varepsilon > 0$ is arbitrary, this will do the trick.

The uniform continuity of f ensures that there is some $\delta > 0$ such that

$$x, y \in D,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}. \tag{7}$$

Also, the definitions of $f(6-)$ and $f(6+)$ imply the existence of two positive reals δ_ℓ and δ_r such that

$$x \in (6 - \delta_\ell, 6) \Rightarrow |f(x) - f(6-)| < \frac{\varepsilon}{3} \tag{8}$$

and similarly

$$y \in (6, 6 + \delta_r) \Rightarrow |f(y) - f(6+)| < \frac{\varepsilon}{3}. \tag{9}$$

Now choose $x < 6$ and $y > 6$ so that all three conditions on the left hand sides of (7), (8) and (9) hold. That is,

$$x > 6 - \delta_\ell,\ y < 6 + \delta_r \text{ and } |x - y| < \delta.$$

Then, the three inequalities on the right hand sides of (7), (8) and (9) hold as well, and by the triangle inequality we get

$$|f(6-) - f(6+)| \le |f(6-) - f(x)| + |f(x) - f(y)| + |f(y) - f(6+)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which is exactly the desired inequality (6). As observed before, this wraps up the proof. ■

E4. Let (X, d) be a metric pace and $E \subseteq X$ a closed subset.

(a) For a point $x \in X$ define its *distance* $d(x, E)$ from the set E by

$$d(x, E) = \inf\{d(x, y) :\ y \in E\}.$$

Show that $d(x, E) = 0$ if and only if $x \in E$.

(b) Show that for every closed subset $E \subseteq X$ there is a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0)$ is exactly E.

(*Hint: yes, you're supposed to figure out how to use part (a) to do part (b).*)

Solution. We do the two parts in turn.

(a) First, observe that $d(x, E)$ is the infimum of a set of non-negative reals, so it is certainly ≥ 0. I will take this for granted from now on.

I have to prove two implications. The easier one goes like this: suppose $x \in E$. Then

$$d(x,E) = \inf\{d(x,y):\ y \in E\} \le d(x,x) = 0.$$

Since, as observed above, we also have $d(x,E) \ge 0$, in this case we get $d(x,E) = 0$ on the nose.

For the other implication, suppose $x \notin E$, meaning that x is contained in the complement E^c. E is closed, so E^c is open (Theorem 2.23). This means that there is a positive real $r > 0$ such that the neighborhood $N_r(x)$ is contained in E^c. In turn, this means that for every $y \in E$ we have $d(x,y) \ge r$, so that the set $\{d(x,y):\ y \in E\}$ is contained in $[r,\infty)$. Its infimum $d(x,E)$ is then also $\ge r > 0$, so we're done.

(b) What the hint is getting at is that you should take $f(x) = d(x,E)$. To show that this works, you now have to prove two things:

First, I want $f^{-1}(0) = E$. This means that $d(x,E) = 0$ *exactly* when $x \in E$, which is nothing but part (a) of the problem!

Secondly, I want to prove that $f : X \to \mathbb{R}$ defined by $f(x) = d(x,E)$ is continuous. We proceed to do this. The goal is proving that if $x_n \to x$ in X then $f(x_n) \to f(x)$. What this translates to is that for every $\varepsilon > 0$ there exists a positive integer N such that

$$|d(x_n,E) - d(x,E)| \le \varepsilon,\ \forall n \ge N. \tag{10}$$

So fix a positive $\varepsilon > 0$. Because $x_n \to x$, I can find N such that $d(x_n,x) < \varepsilon$ for $n \ge N$. But then, using the triangle inequality in X, we get

$$\forall P \in E,\ \forall n \ge N,\ d(x_n,p) \le d(x,p) + d(x_n,x) < d(x,p) + \varepsilon. \tag{11}$$

Similarly, reversing the roles of x_n and x, we have

$$\forall p \in E,\ \forall n \ge N,\ d(x,p) \le d(x_n,p) + d(x_n,x) < d(x_n,p) + \varepsilon. \tag{12}$$

Now, (11) implies

$$d(x_n,E) = \inf_{y\in E} d(x_n,y) \le d(x_n,p) < d(x,p) + \varepsilon$$

for all $p \in E$ and all $n \ge N$. Putting the ε on the other side this reads

$$d(x_n,E) - \varepsilon < d(x,p),\ \forall p \in E,\ \forall n \ge N,$$

so for all $n \ge N$ the number $d(x_n,E) - \varepsilon$ is a lower bound for the set $\{d(x,p):\ p \in E\}$. But the infimum $d(x,E)$ is the *greatest* lower bound of this set, so we have

$$d(x_n,E) - \varepsilon \le d(x,E). \tag{13}$$

Similarly, interchanging the roles of x_n and x again and using (12), we get

$$d(x,E) - \varepsilon \le d(x_n,E). \tag{14}$$

Equations (12) and (13) hold for all $n \ge N$, so for all such n we have (10). This finishes the proof. ■