Math 424 B / 574 B Due Wednesday, Dec 09
Autumn 2015

Homework 9

This assignment is meant to cover material from multiple chapters and act as a kind of review. That's not to say it covers *everything* on the test, because of space limitations, but it's a good selection.

Please do Problems 1 and 4 from Chapter 3 (page 78), plus the additional problems below. In Problem 1 from the textbook assume that the terms of the sequence are either real numbers or complex numbers (whichever you prefer; I will consider both as correct).

We need the following concept.

Definition 1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is *bounded* if $f(X)$ is a bounded subset of Y in the sense of Definition 2.18 (i) in our textbook. ♦

E1. Let (X, d_X) be a metric space, and let $\mathcal{B}(X)$ be the set of all functions from X to $\mathbb{R}$ that are bounded in the sense of Definition 1. For $f, g \in \mathcal{B}(X)$ define

$$d(f, g) = \sup\{|f(x) - g(x)| : \ x \in X\}.$$

(a) Show that $d(f, g)$ is a non-negative number for all $f, g \in \mathcal{B}(X)$ (that is, it's never ∞).
(b) Show that $\mathcal{B}(X)$ equipped with d is a metric space.
(c) Prove that the subset $\mathcal{CB}(X) \subset \mathcal{B}(X)$ consisting of those bounded functions $X \to \mathbb{R}$ which are also continuous is closed in the metric space $(\mathcal{B}(X), d)$ from part (b).
(d) Show that the metric space $(\mathcal{B}(X), d)$ is complete.

For the next problem, recall

Definition 2. A point x of some metric space (X, d) is called a *condensation point* of a subset $E \subseteq X$ if every neighborhood of x contains uncountably many points of E. ♦

The definition appears in Problem 27 from page 45 of the textbook.

E2. Let $E \subset \mathbb{R}$ be an uncountable subset. Show that E contains one of its condensation points.

In the following problem regard $(-2, 6) \cup (6, 10)$ as a metric space with the usual metric inherited from $\mathbb{R}$.

E3. Is there a continuous function $f : (-2, 6) \cup (6, 10) \to \mathbb{R}$ for which $f(6-)$ and $f(6+)$ both exist and such that $f(6-) < f(6+)$? How about a *uniformly* continuous one?

E4. Let (X, d) be a metric pace and $E \subseteq X$ a closed subset.

(a) For a point $x \in X$ define its *distance* $d(x, E)$ from the set E by
$$d(x, E) = \inf\{d(x, y) : \; y \in E\}.$$
Show that $d(x, E) = 0$ if and only if $x \in E$.

(b) Show that for every closed subset $E \subseteq X$ there is a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0)$ is exactly E.

(*Hint: yes, you're supposed to figure out how to use part (a) to do part (b).*)