Math 424 B / 574 B Due Wednesday, Dec 02
Autumn 2015

Solutions for Homework 8

Solution for Problem 17, page 100. I will give a proof that does not follow the hint in the book.

Denote by $S \subseteq (a, b)$ the set of simple discontinuities of f. A point $x \in (a, b)$ is a simple discontinuity if and only if $f(x+)$ and $f(x-)$ both exist and $f(x-)$, $f(x)$ and $f(x+)$ are *not* all equal, S is the union of the sets

$$S^+ = \{x \in S : \; f(x) \neq f(x+)\}$$

and

$$S^- = \{x \in S : \; f(x) \neq f(x-)\}.$$

This means that it suffices to show that both S^+ and S^- are at most countable. I will do this for S^+, since the argument for S^- works in exactly the same way.

Now, if the two numbers $f(x)$ and $f(x+)$ are different, the absolute value $|f(x) - f(x+)|$ is $\geq \frac{1}{n}$ for *some* positive integer n. This means that we have

$$S^+ = \bigcup_{n=1}^{\infty} S_n^+,$$

where

$$S_n^+ = \left\{ x \in S^+ : \; |f(x) - f(x+)| \geq \frac{1}{n} \right\}.$$

The union of countably many sets that are at most countable is at most countable (by the corollary to Theorem 2.12 from the book), so it's enough to show that each S_n^+ is at most countable. This is what we set out to prove now.

The proof proceeds by contradiction: suppose that for some positive integer n the set S_n^+ is uncountable. Then, some point $x \in S_n^+$ is a limit point of S_n^+ (take this as an exercise; Problem E2 from your next homework asks you to prove that for an uncountable set E of real numbers, some element of E).

In conclusion, we can find a sequence $(x_m)_m$ in $S_n^+ \setminus \{x\}$ that converges to x. Moreover, since either infinitely many x_m are bigger than x or infinitely many of them are smaller than x, we can assume only one of those things is true (that is, either $x_m > x$ for all m or $x_m < x$ for all m). The two assumptions will lead to parallel proofs, so without loss of generality assume that $x_m > x$ for all m.

Now, we have $x_m \to x$ from above, so by the definition of $f(x+)$ (and since we know this number exists) we have

$$f(x_m) \to f(x+). \tag{1}$$

On the other hand, each x_m belongs to S_n^+, so by the definition of this set we can find $y_m > x$ with

$$x_m < y_m < x_m + \frac{1}{n} \tag{2}$$

such that $|f(x_m) - f(y_m)| \geq \frac{1}{n}$.

Now, on the one hand, (2) means that $y_m \to x$ from above and hence

$$f(y_m) \to f(x+). \tag{3}$$

On the other hand though, $|f(x_m) - f(y_m)| \geq \frac{1}{n}$ and (1) together imply

$$|f(y_m) - f(x+)| \geq \frac{1}{n}. \tag{4}$$

Clearly, (3) and (4) are mutually contradictory, so we have our contradiction and are done. ■

Throughout this assignment the letter I denotes an interval in $\mathbb{R}$ in the sense of Definition 2.17 in your book, i.e. a set of the form $[a, b]$ for some $a < b \in \mathbb{R}$. The *interior* of an interval $[a, b]$ means everything in the interval except for its endpoints, i.e. (a, b).

I will drop the term 'monotonically' from 'monotonically increasing' and 'monotonically decreasing' for brevity.

For the first problem, I'll need the following notion.

Definition 1. Let $f : X \to \mathbb{R}$ be a function defined on a metric space (X, d).

A *local maximum* for f is a point $x \in X$ for which there is some $\delta > 0$ such that

$$f(x') \leq f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \leq \delta.$$

(in other words, the largest value of f in the δ-neighborhood of x is achieved at x).

Similarly, a *local minimum* of f is a point $x \in X$ such that

$$f(x') \geq f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \leq \delta.$$

(i.e. the smallest value of f in the δ-neighborhood of x is achieved at x). ♦

With this in place, the first problem reads as follows.

E1. Show that if $f : I \to \mathbb{R}$ is continuous and has no local maxima or minima in the interior of I, then it is monotonic.

Before going into the proof of this, I will prove an auxiliary result that helps with both Problem E1 and Problem E2.

Lemma 2. *If $f : I \to \mathbb{R}$ is not monotonic, then there are points $x < y < z$ such that either*

$$f(x) < f(y) > f(z) \tag{5}$$

or

$$f(x) > f(y) < f(z). \tag{6}$$

Proof. Suppose f is not monotonic. This means that we can find $p < q \in I$ such that $f(p) < f(q)$ and also $u < v \in I$ with $f(u) > f(v)$. Assume without loss of generality that $q \leq v$; if not, we can simply interchange the roles of p and u, the roles of q and v, and switch from f to $-f$.

We further consider three cases.

Case 1: $q \le u$**.** So the ordering is $p < q \le u < v$. If $f(q) \le f(u)$ then you can take $x = p$, $y = u$ and $z = v$ and you've got (5). On the other hand, if $f(q) > f(u)$ take $x = p$, $y = q$ and $z = v$ and again you have (5).

Case 2: $u \le p$**.** Our four numbers are ordered as $u \le p < q \le v$. If $f(u) \le f(p)$ take $x = p$, $y = q$ and $z = v$ and you have (7). Otherwise, if $f(u) > f(p)$ set $x = u$, $y = p$ and $z = q$, and you get (6).

Case 3: $u \in (p, q)$**.** The ordering of the four numbers is now $p < u < q \le v$. If $f(u)$ is outside the interval $[f(p), f(q)]$ then set $x = p$, $y = u$ and $z = q$ and you have either (5) or (6). If $f(p) \le f(u) \le f(q)$ then $f(v) < f(u) \le f(q)$ and setting $x = p$, $y = q$ and $z = v$ gets you (5). ■

With this in our toolkit we can give the

Solution to Problem E1. The proof is by contradiction, assuming f satisfies the conditions in the statement and is *not* monotonic. We can then apply Lemma 2. To fix ideas, I will assume throughout this proof that we have $x < y < z$ such that (5) holds; the other case is entirely analogous.

Since $f(y)$ is larger than both $f(x)$ and $f(z)$, neither $f(x)$ nor $f(z)$ can be the maximum value of f over $[x, z]$. But there is *some* $t \in [x, z]$ where the restriction of f to $[x, z]$ reaches its maximum, by Theorem 4.16 (because $[x, z]$ is compact).

The number t must be in the interior of $[x, z]$ (since it's neither x nor z, as just observed), and is hence in the interior of I. It is also a local maximum for f because it is a maximum of f restricted to $[x, z]$. This contradicts the assumption that f has no local maxima, and we are done. ■

For the next two problems I need

Definition 3. A real-valued function $f : I \to \mathbb{R}$ is *strictly increasing* if for $x < y$ in I we have $f(x) < f(y)$.

f is *strictly decreasing* if

$$x < y \Rightarrow f(x) > f(y).$$

f is *strictly monotonic* if it is either strictly increasing or strictly decreasing. ♦

E2. Show that if $f : I \to \mathbb{R}$ is continuous and one-to-one then it is strictly monotonic.

Solution. As for E1, we prove this by contradiction, assuming f is continuous, one-to-one, and not strictly monotonic.

I will use a version of Lemma 2 adapted to the strictly monotonic case, saying that if f is not strictly monotonic then there are $x < y < z \in I$ such that one of (5) and (6) holds with non-strict inequality signs (I'll leave this version of Lemma 2 as an exercise). The injectivity of f, however, ensures that the signs *are* strict.

In conclusion, if the one-to-one function f is *not* strictly monotonic, then there are $x < y < z$ in I satisfying either (5) or (6). And as in the proof of E1 we assume (5) holds, the other case being analogous.

For sufficiently small $\varepsilon > 0$ the number $f(y) - \varepsilon$ is contained in both $(f(x), f(y))$ and $(f(z), f(y))$. But then, Theorem 4.23 ensures that there are points

$$t_1 \in (x, y),\ t_2 \in (y, z)$$

such that $f(t_1) = f(t_2) = f(y) - \varepsilon$. This contradicts the injectivity of f and finishes the proof. ■

E3. Let $f : I \to \mathbb{R}$ be a strictly increasing function. Show that if the image of f is (a) connected or (b) closed then f is continuous.

Solution. Suppose f is not continuous. According to Theorem 4.29, this means that there is some simple discontinuity $x \in I$. We need to derive a contradiction from this.

Now, either x is one of the endpoints a, b of $I = [a, b]$ or x is in the interior (a, b). I will just treat the latter case; $x = a$ or $x = b$ are similar (see Remark 4 below).

We know that $x \in (a, b)$ is a simple discontinuity, so $f(x-)$ and $f(x+)$ both exist, but $f(x-) < f(x+)$ (otherwise they'd be equal to $f(x)$, since by Theorem 4.29 this number is between $f(x-)$ and $f(x+)$).

Now suppose we're in case (a) of the problem, meaning that the image $f(I)$ is connected. Then, according to Theorem 2.47, *every* number between $f(a)$ and $f(b)$ is contained in $f(I)$. Theorem 4.29, however, says in our case that

$$\sup_{a \le t < x} f(t) = f(x-) < f(x+) = \inf_{x < t \le b} f(t), \tag{7}$$

so the interval $(f(x-), f(x+))$ is not contained in $f(I)$ except perhaps for the single point $f(x)$; this is a contradiction, and we're done.

Assume now that we are in case (b) and know that $f(I)$ is closed. By (7) both $f(x-)$ and $f(x+)$ are contained in $f(I)$ (because the supremum of a set that is bounded above is in the closure of that set, and similarly for the infimum of a set bounded below; see Theorem 2.28).

(7) shows that $f(x+)$ is strictly less than all $f(t)$, $x < t \le b$ and strictly greater than all $f(t)$, $a \le t < x$. The only possibility then is that $f(x+) = f(x)$. Similarly, $f(x-) = f(x)$, which means that

$$f(x-) = f(x) = f(x+)$$

and contradicts our assumption that f is *not* continuous at x. ■

Remark 4. The case $x = a$ of Problem E3 is treated similarly, but is easier; here, only $f(x+)$ makes sense (there are no $t < x$ in $I = [a, b]$), and the discontinuity assumption means that $f(x) < f(x+)$.

Similarly, in the case $x = b$ only $f(x-)$ makes sense and the assumption is $f(x-) < f(x)$. I will leave these as exercises; the arguments are analogous to the one for $x \in (a, b)$ presented here. ♦

E4. Suppose the function $f : I \to \mathbb{R}$ has the following property:
Each $x \in I$ has a neighborhood $N_{r(x)}(x)$ in I on which f is increasing.
Show that f is increasing.

Here's a bit of a discussion regarding this last problem.

My notation $N_{r(x)}(x)$ is meant to indicate that the radius $r(x)$ of the neighborhood might depend on x (so for some points x you know f to be increasing only on tiny neighborhoods around x).

You have to show that for any two $x_1 < x_2 \in I$ you have $f(x_1) \leq f(x_2)$. Now, as x ranges over the closed interval $[x_1, x_2]$, the neighborhoods $N_{\frac{r(x)}{2}}(x)$ form an open cover of that interval. Try to see what happens after you extract a finite subcover from it.

Solution. As the hint suggests, fix two arbitrary points $x_1 < x_2$ in I and cover the interval $J = [x_1, x_2]$ with the neighborhoods

$$N_{\frac{r(x)}{2}}(x) = I \cap \left(x - \frac{r(x)}{2}, x + \frac{r(x)}{2} \right)$$

for $x \in J$. As J is compact, we can extract from this a finite subcover. That means that there are finitely many points y_1 up to y_n in J such that $N_i = N_{\frac{y_i}{2}}(y_i)$ cover J for $1 \leq i \leq n$.

Assume we've indexed the y_i so that $y_1 < y_2 < \cdots < y_n$ (if not you can always just reindex them). The condition that N_i cover J ensures that $x_1 \in N_1$, so $f(x_1) \leq f(y_1)$ because the restriction of f to N_1 is increasing.

Similarly, at the other end, we have $f(y_n) \leq f(x_2)$.

As for what happens in between, for every $1 \leq i \leq n-1$ the neighborhoods N_i and N_{i+1} intersect at some point y, and since the restriction of f to both N_i and N_{i+1} is increasing we have

$$f(y_i) \leq f(y) \leq f(y_{i+1}).$$

All in all we have

$$f(x_1) \leq f(y_1) \leq f(y_2) \leq \cdots \leq f(y_n) \leq f(x_2).$$

Since $x_1 < x_2$ were chosen arbitrarily, this finishes the proof. ■