Math 424 B / 574 B Due Wednesday, Dec 02
Autumn 2015

Homework 8

Problem 17 from page 100 of our book, plus the four problems below.

Throughout this assignment the letter I denotes an interval in $\mathbb{R}$ in the sense of Definition 2.17 in your book, i.e. a set of the form $[a, b]$ for some $a < b \in \mathbb{R}$. The *interior* of an interval $[a, b]$ means everything in the interval except for its endpoints, i.e. (a, b).

I will drop the term 'monotonically' from 'monotonically increasing' and 'monotonically decreasing' for brevity.

For the first problem, I'll need the following notion.

Definition 1. Let $f : X \to \mathbb{R}$ be a function defined on a metric space (X, d).

A *local maximum* for f is a point $x \in X$ for which there is some $\delta > 0$ such that

$$f(x') \le f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \le \delta.$$

(in other words, the largest value of f in the δ-neighborhood of x is achieved at x).

Similarly, a *local minimum* of f is a point $x \in X$ such that

$$f(x') \ge f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \le \delta.$$

(i.e. the smallest value of f in the δ-neighborhood of x is achieved at x). ♦

With this in place, the first problem reads as follows.

E1. Show that if $f : I \to \mathbb{R}$ is continuous and has no local maxima or minima in the interior of I, then it is monotonic.

For the next two problems I need

Definition 2. A real-valued function $f : I \to \mathbb{R}$ is *strictly increasing* if for $x < y$ in I we have $f(x) < f(y)$.

f is *strictly decreasing* if

$$x < y \Rightarrow f(x) > f(y).$$

f is *strictly monotonic* if it is either strictly increasing or strictly decreasing. ♦

E2. Show that if $f : I \to \mathbb{R}$ is continuous and one-to-one then it is strictly monotonic.

E3. Let $f : I \to \mathbb{R}$ be a strictly increasing function. Show that if the image of f is (a) connected or (b) closed then f is continuous.

E4. Suppose the function $f : I \to \mathbb{R}$ has the following property:
Each $x \in I$ has a neighborhood $N_{r(x)}(x)$ in I on which f is increasing.
Show that f is increasing.

Here's a bit of a discussion regarding this last problem.

My notation $N_{r(x)}(x)$ is meant to indicate that the radius $r(x)$ of the neighborhood might depend on x (so for some points x you know f to be increasing only on tiny neighborhoods around x).

You have to show that for any two $x_1 < x_2 \in I$ you have $f(x_1) \leq f(x_2)$. Now, as x ranges over the closed interval $[x_1, x_2]$, the neighborhoods $N_{\frac{r(x)}{2}}(x)$ form an open cover of that interval. Try to see what happens after you extract a finite subcover from it.