Homework 8

Problem 17 from page 100 of our book, plus the four problems below.

Throughout this assignment the letter \( I \) denotes an interval in \( \mathbb{R} \) in the sense of Definition 2.17 in your book, i.e. a set of the form \([a, b]\) for some \(a < b \in \mathbb{R}\). The interior of an interval \([a, b]\) means everything in the interval except for its endpoints, i.e. \((a, b)\).

I will drop the term 'monotonically' from 'monotonically increasing' and 'monotonically decreasing' for brevity.

For the first problem, I'll need the following notion.

**Definition 1.** Let \( f : X \to \mathbb{R} \) be a function defined on a metric space \((X, d)\).

A local maximum for \( f \) is a point \( x \in X \) for which there is some \( \delta > 0 \) such that

\[
f(x') \leq f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \leq \delta.
\]

(in other words, the largest value of \( f \) in the \( \delta \)-neighborhood of \( x \) is achieved at \( x \)).

Similarly, a local minimum of \( f \) is a point \( x \in X \) such that

\[
f(x') \geq f(x) \quad \text{for all} \quad x' \in X \quad \text{such that} \quad d(x', x) \leq \delta.
\]

(i.e. the smallest value of \( f \) in the \( \delta \)-neighborhood of \( x \) is achieved at \( x \)).

With this in place, the first problem reads as follows.

**E1.** Show that if \( f : I \to \mathbb{R} \) is continuous and has no local maxima or minima in the interior of \( I \), then it is monotonic.

For the next two problems I need

**Definition 2.** A real-valued function \( f : I \to \mathbb{R} \) is **strictly increasing** if for \( x < y \) in \( I \) we have \( f(x) < f(y) \).

\( f \) is **strictly decreasing** if

\[
x < y \Rightarrow f(x) > f(y).
\]

\( f \) is **strictly monotonic** if it is either strictly increasing or strictly decreasing.

**E2.** Show that if \( f : I \to \mathbb{R} \) is continuous and one-to-one then it is strictly monotonic.

**E3.** Let \( f : I \to \mathbb{R} \) be a strictly increasing function. Show that if the image of \( f \) is (a) connected or (b) closed then \( f \) is continuous.
E4. Suppose the function $f : I \to \mathbb{R}$ has the following property:
Each $x \in I$ has a neighborhood $N_{r(x)}(x)$ in $I$ on which $f$ is increasing.
Show that $f$ is increasing.

Here’s a bit of a discussion regarding this last problem.
My notation $N_{r(x)}(x)$ is meant to indicate that the radius $r(x)$ of the neighborhood might
depend on $x$ (so for some points $x$ you know $f$ to be increasing only on tiny neighborhoods
around $x$).

You have to show that for any two $x_1 < x_2 \in I$ you have $f(x_1) \leq f(x_2)$. Now, as $x$
ranges over the closed interval $[x_1, x_2]$, the neighborhoods $N_{r(x)}(x)$ form an open cover of
that interval. Try to see what happens after you extract a finite subcover from it.