**Math 424 B / 574 B** Due Wednesday, Nov 25
Autumn 2015

# Solutions for Homework 7

**Solution for Problem 6, page 99.** I'll follow the hint from the assignment and work with the function $F : E \to \mathbb{R}^2$ defined by

$$F(x) = (x, f(x)).$$

($\Rightarrow$) Suppose $f$ is continuous. Then, since $x \mapsto x$ is continuous, $F$ is continuous by part (a) of Theorem 4.10 (because its components $x \mapsto x$ and $f$ are continuous).

So the graph

$$\Gamma(f) = \{(x, f(x)) : \ x \in E\}$$

of $f$ is the image of $E$ through the continuous function $F$. Since continuous functions preserve compactness (Theorem 4.14), the graph is indeed compact.

($\Leftarrow$) Conversely, assume that the graph $\Gamma(f)$ is compact. We want to show that $f$ is continuous, or equivalently (by the corollary to Theorem 4.8) that the inverse image $f^{-1}(C)$ of a closed subset $C \subseteq R$ is closed.

$f^{-1}(C)$ consists of those points $x \in E$ such that $f(x) \in C$, or equivalently, such that the second component of $F(x) = (x, f(x))$ belongs to $C$. In other words, $f^{-1}(C)$ consists of the first components of those points $(x, f(x)) \in \Gamma(f)$ which also belong to $\mathbb{R} \times C$. If I denote the two coordinate functions on $\mathbb{R}^2$ by $\phi_1$ and $\phi_2$ as in Example 4.11, so that

$$\phi_1(x_1, x_2) = x_1 \quad \text{and} \quad \phi_2(x_1, x_2) = x_2,$$

then we've just proven the following formula:

$$f^{-1}(C) = \phi_1\left(\Gamma(f) \cap \phi_2^{-1}(C)\right). \tag{1}$$

$C \subseteq \mathbb{R}$ is closed, so $\phi_2^{-1}(C) \subseteq \mathbb{R}^2$ is closed because the coordinate functions $\phi_i$ are continuous (Example 4.11).

We are assuming that $\Gamma(f)$ is compact, so $\Gamma(f) \cap \phi_2^{-1}(C)$ is again compact, being a closed subset of a compact metric space (Theorem 2.35).

Finally, $\phi_1 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, so it maps compact sets onto compact sets by Theorem 4.14. All in all, (1) now ensures that $f^{-1}(C) \subset E \subset \mathbb{R}$ is compact and hence closed by Theorem 2.34. ■

**Solution for Problem 8, page 99.** By uniform continuity there is a $\delta > 0$ such that

$$|f(x) - f(y)| < 1 \quad \text{provided} \quad x, y \in E \quad \text{and} \quad |x - y| < \delta. \tag{2}$$

Since $E \subset \mathbb{R}$ is bounded, it can be covered by finitely many open intervals $I_1$ up to $I_n$ of length $\delta$. We then get

$$f(E) = f\left(\bigcup_{j=1}^{n}(E \cap I_j)\right) \subseteq \bigcup_{j=1}^{n} f(E \cap I_j).$$

(2) means that each $f(E \cap I_j)$ is bounded (because it has diameter $\leq 1$), so that $f(E)$ is contained in a finite union of bounded sets. Finite unions of bounded sets are bounded, so we are done.

To see that the boundedness of $E$ is essential, take say $E = \mathbb{R}$ and define $f$ by $f(x) = x$ for all $x \in E$. It is immediate that $f$ is uniformly bounded, but the range of $f$ is all of $R$ which is unbounded. ■

**Solution for Problem 11, page 99.** I will give the alternate proof for Problem 13 separately, after this one. Here, I'm just showing that $(f(x_n))_n$ is Cauchy if $f : X \to Y$ is uniformly continuous and $(x_n)_n$ is Cauchy.

Fix $\varepsilon > 0$. We want to show that there is some positive integer $N$ such that

$$n, m \geq N \Rightarrow d(f(x_n), f(x_m)) < \varepsilon. \tag{3}$$

Now, by uniform continuity, we know that there is some $\delta > 0$ such that

$$d(x_n, x_m) < \delta \Rightarrow d(f(x_n), f(x_m)) < \varepsilon. \tag{4}$$

On the other hand, because $(x_n)_n$ is Cauchy, there is a positive integer $N$ such that

$$n, m \geq N \Rightarrow d(x_n, x_m) < \delta. \tag{5}$$

Equations (4) and (5) together give (3), which completes the proof. ■

**Alternate solution for Problem 13, page 99.** Let $x \in X \setminus E$ be an arbitrary point. First off, we want to define the value $g(x)$ of an extension $g : X \to \mathbb{R}$ of $f$ (this is what it means to *extend* $f$ from $E$ to all of $X$; see Problem 5 from the textbook for the definition of 'extension').

Since $E$ is dense, there is a sequence $(x_n)_n$ in $E$ that converges to $x$. Since $x_n \to x$ in $X$, the sequence $(x_n)_n$ is Cauchy. By Problem 11 solved above the absolute continuity of $f$ implies that $(f(x_n))_n$ is also Cauchy. Since by part (c) of Theorem 3.11 the real line $\mathbb{R}$ is complete, the Cauchy sequence $(f(x_n))_n$ converges to some $y \in \mathbb{R}$; set $g(x) = y$. We now have to check a few things.

**Step 1: The value $g(x) = y$ from above is well defined.** Here's what this means: Recall that in order to set $g(x) = y$ I *chose* some sequence $(x_n)_n \subset E$ converging to $x$. What I want to prove now is that the real number $y$ from the construction above does not depend on the specific sequence $x_n$ that converges to $x$, only on $x$ itself (otherwise the formula $g(x) = y$ is not justified).

Suppose you have sequences $x_n \to x$ and $x'_n \to x$ with $x_n, x'_n \in E$ and fix an arbitrarily small $\varepsilon > 0$. Since $f$ is uniformly continuous, there is some $\delta > 0$ such that

$$d(p, q) < \delta \Rightarrow d(f(p), f(q)) < \varepsilon. \tag{6}$$

For sufficiently large $n$ we have

$$d(x_n, x) < \frac{\delta}{2},\ d(x'_n, x) < \frac{\delta}{2},$$

which implies

$$d(x_n, x'_n) \le d(x_n, x) + d(x'_n, x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

This together with (6) implies $d(f(x_n), f(x'_n)) < \varepsilon$ for sufficiently large $n$, so the distance between the limits of the Cauchy sequences $(f(x_n))_n$ and $(f(x'_n))_n$ is at most $\varepsilon$ (this last step follows for example from Problem 6 from Homework 4). This is true for *any* $\varepsilon > 0$, so the two sequences $(f(x_n))_n$ and $(f(x'_n))_n$ must have the same limit. So indeed $y = \lim_n f(x_n)$ does not depend on $(x_n)_n$ itself so long as this sequence converges to $x$.

This concludes the proof of Step 1. Since I now have a well defined value for $g$ at each point in $X \setminus E$. Setting $g = f$ on $E$ gives me a function $g : X \to \mathbb{R}$ which is an extension of $f$ in the sense of Problem 5. We want a *continuous* extension, so there's one more thing to check:

**Step 2: The function $g : X \to \mathbb{R}$ defined above is continuous.** We will actually show that $g$ is *uniformly* continuous, which is even better.

Note first that the exact same procedure as above actually works for $x \in E$ as well, so we can define $g$ uniformly for all points of $X$: pick your point $x \in X$ arbitrarily, take some sequence $E \ni x_n \to x$ converging to it, and set $g(x)$ to be the limit of $f(x_n)$. This observation will allow us to treat all points of $X$ on an equal footing, without having to split into cases depending on whether or not they are in $E$.

Fix $\varepsilon > 0$. We want to find some $\delta > 0$ such that

$$d(p, x) < \delta \Rightarrow d(g(p), g(x)) < \varepsilon.$$

Choose $\delta > 0$ so small that

$$p', x' \in E,\ d(p', x') < 3\delta \Rightarrow d(f(p'), f(x')) < \frac{\varepsilon}{3}. \tag{7}$$

(this is possible, since $f : E \to \mathbb{R}$ is uniformly continuous).

Now, by the way we defined $g$, we can find $x' \in E$ such that $d(x', x) < \delta$ and $d(f(x'), g(x)) < \frac{\varepsilon}{3}$.

Similarly, for any $p \in X$ with $d(p, x) < \delta$ we can find $p' \in E$ with $d(p', p) < \delta$ and $d(f(p'), g(p)) < \frac{\varepsilon}{3}$.

We then have

$$d(p', x') \le d(p', p) + d(p, x) + d(x', x) < \delta + \delta + \delta = 3\delta,$$

which by (7) implies $d(f(p'), f(x')) < \frac{\varepsilon}{3}$. But then we get

$$d(g(p), g(x)) \le d(f(p'), g(p)) + d(f(p'), f(x')) + d(f(x'), g(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which is what we wanted. ■

**Remark 1.** Problem 13 also asks whether the result remains true if instead of real-valued functions $f$ we consider functions taking values in any Euclidean space $\mathbb{R}^k$, or any compact or complete or arbitrary metric space. I purposely phrased the proof so that it goes through whenever $f : E \to Y$ lands inside a *complete* metric space $Y$.

You can't drop completeness though: the identity function $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}$ cannot be extended to a continuous function $\mathbb{R} \to \mathbb{Q}$. Indeed, if it could, the image of this continuous function would contain both 0 and 1 and hence would have to contain the entire interval $[0,1]$ by Theorem 4.23. This means that its image can't be contained in $\mathbb{Q}$.

In conclusion, the answer to all of the questions at the end of Problem 13 is 'yes' except for the last one, which has a negative answer. ♦

**E1.** Give an example of a continuous function $f:(0,1)\to\mathbb{R}$ and a Cauchy sequence $(x_n)_n$ in $(0,1)$ such that $(f(x_n))_n$ is not Cauchy in $\mathbb{R}$.

**Solution.** Take say $f(x)=\frac{1}{x}$. This is continuous by, say, Theorem 4.9 (dividing a continuous function by a non-zero continuous functions yields something continuous).

Now have a look at the sequence $\frac{1}{n}$ in $(0,1)$. It is Cauchy because it is convergent in the larger metric space $[0,1)$, but applying $f$ to it turns it into the sequence $n$, which is unbounded and hence not Cauchy. ■

**E2.** Show that a metric space $(X,d)$ is disconnected if and only if there exists a continuous function $f:X\to\mathbb{R}$ whose range $f(X)$ is the two-element set $\{0,1\}$.

**Solution.** According to Problem 1 from Homework 4, a metric space is disconnected if and only if it can be written as the disjoint union of two non-empty open subsets (or equivalently, two non-empty *closed* subsets); see also Definition 3 in that homework assignment.

($\Rightarrow$) Suppose first that I can write $X\ A\cup B$, with $A\cap B=\emptyset$ and both $A$ and $B$ are open and non-empty. Define a function $f:X\to\mathbb{R}$ by

$$f(x)=\begin{cases}0 & \text{if } x\in A\\ 1 & \text{if } x\in B\end{cases}$$

Clearly, $f(X)=\{0,1\}$. To see that $f$ is continuous, use the criterion that the preimage of every relatively open subset of $\{0,1\}\subset\mathbb{R}$ is open. $\{0,1\}$ has four subsets, namely

$$\emptyset,\ \{0\},\ \{1\} \text{ and } \{0,1\}.$$

Their preimages through $f$ are

$$\emptyset,\ A,\ B \text{ and respectively } X,$$

all of which are indeed open.

($\Leftarrow$) If there is a function $f:X\to\mathbb{R}$ as in the statement, then the inverse images $A=f^{-1}(0)$ and $B=f^{-1}(1)$ of the relatively open subsets

$$\{0\} \text{ and } \{1\}\subset\{0,1\}$$

are open and non-empty in $X$, and moreover

$$X=f^{-1}(\{0,1\})=f^{-1}(0)\cup f^{-1}(1)=A\cup B,$$

so we can write $X$ as a disjoint union of two non-empty open subsets. According to the observation made at the beginning of the proof, this means that $X$ is disconnected. ■