**Math 424 B / 574 B** Due Wednesday, Nov 18
Autumn 2015

# Solutions for Homework 6

**Solution for Problem 5, page 78.** For convenience, throughout this proof I will use the phrase 'subsequential limit' to also possibly refer to $\pm\infty$ if some subsequence of the sequence under consideration goes to $\pm\infty$ in the sense of Definition 3.15 from your textbook.

We break up the problem into several cases.

**Case 1: $\overline{\lim}\ a_n = \infty$ or $\overline{\lim}\ b_n = \infty$.**

Then, the conclusion is immediate: by assumption $\overline{\lim}\ a_n + \overline{\lim}\ b_n$ makes sense as addition in the extended real number line, so it must be $\infty$ (the arithmetic rules for adding extended real numbers are reviewed on page 12 of our textbook). But then the inequality

$$\overline{\lim}(a_n + b_n) \leq \overline{\lim}\ a_n + \overline{\lim}\ b_n = \infty$$

is clear.

**Case 2: $\overline{\lim}\ a_n$ and $\overline{\lim}\ b_n$ are both $< \infty$, but at least one is $-\infty$.** Assume without loss of generality that $\overline{\lim}\ a_n = -\infty$ (if the other one is $-\infty$ we can just change labels, interchanging $a$s and $b$s).

Now, we are assuming $\overline{\lim}\ b_n < \infty$, so the sequence $(b_n)_n$ is bounded above, say $b_n \leq M$ for some real number $M$. Since $\overline{\lim}\ a_n = -\infty$, we have $a_n \to -\infty$. But then $b_n \leq M$ implies $a_n + b_n \to -\infty$, and so $\overline{\lim}(a_n + b_n) = -\infty$. This means that once more we get

$$-\infty = \overline{\lim}(a_n + b_n) \leq \overline{\lim}\ a_n + \overline{\lim}\ b_n.$$

**Case 3: $\overline{\lim}\ a_n$ and $\overline{\lim}\ b_n$ are both real numbers.** In particular, the sequences $(a_n)_n$ and $(b_n)_n$ are bounded.

Denote $a^* = \overline{\lim}\ a_n$ and similarly, $b^* = \overline{\lim}\ b_n$. By definition, $\overline{\lim}(a_n + b_n)$ is the supremum of the set of subsequential limits of the sequence $(a_n+b_n)_n$. If we show that every subsequential limit of $(a_n + b_n)_n$ is less than or equal to $a^* + b^*$ we will be done, for then $a^* + b^*$ will be an upper bound for the set of subsequential limits of $(a_n + b_n)_n$, whereas $\overline{\lim}(a_n + b_n)$ is the *least* upper bound.

So we are left having to prove that if the subsequence $(a_{n_k} + b_{n_k})_k$ converges to $L$, then $L \leq a^* + b^*$.

Since $(a_{n_k})_k$ and $(b_{n_k})_k$ are bounded by assumption, we can extract from them two convergent subsequences $\left(a_{n_{k_\ell}}\right)_\ell$ and $\left(b_{n_{k_\ell}}\right)_\ell$ respectively by Theorem 3.6 (b). Denoting

$$a = \lim_{\ell}\ a_{n_{k_\ell}} \quad \text{and} \quad b = \lim_{\ell}\ b_{n_{k_\ell}},$$

we have on the one hand $a+b = L$, and on the other hand $a \leq a^*$ and $b \leq b^*$ (because $a^*$ is an upper bound for the set of subsequential limits of $(a_n)_n$, and similarly for $b$). In conclusion, we get

$$L = a + b \leq a^* + b^*,$$

as desired.

To show that the inequality

$$\overline{\lim}(a_n + b_n) \le \overline{\lim}\ a_n + \overline{\lim}\ b_n \tag{1}$$

can be strict, take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. In other words, $a_n$ and $b_n$ both alternate between 1 and $-1$, but they're always opposite in sign. We have $a_n + b_n = 0$ for all $n$, so the left hand side of (1) is 0. On the other hand, both terms of the right hand side of (1) are equal to 1, so the whole right hand side is $2 > 0$. ■

**Solution for Problem 21, page 82.** Since $E_n$ are non-empty, we can find points $x_n \in E_n$. I claim first that the sequence $x_n$ converges to some $x \in \bigcap_n E_n$.

To see this, let us first observe that the sequence $(x_n)_n$ is Cauchy. Indeed, for any $\varepsilon > 0$ there is some $N$ such that diam $E_n < \varepsilon$ for all $n \ge N$. But then, since $x_n \in E_n$, we get

$$d(x_n, x_m) < \varepsilon,\ \forall n, m \ge N,$$

which means that $(x_n)_n$ meets the requirements for being Cauchy. Now, since we are in a complete metric space, Cauchy implies convergent, so that $x_n \to x$ for some $x$.

This just shows that $(x_n)_n$ is convergent. To see that the limit $x$ is in $\bigcap_n E_n$, note that for any positive integer $N$ the terms $x_n$, $n \ge N$ of our sequence are all contained in the closed set $E_N$. Since $x$ is also the limit of the sequence $(x_n)_{n \ge N}$ (because dropping a finite number of terms of a sequence does not alter the limit), $x$ must be contained in $E_N$. This is true for arbitrary $N$, so indeed $x \in \bigcap_n E_n$.

The above discussion shows that $\bigcap_n E_n$ is non-empty (because it contains a point $x$ constructed as above). Now, to see that the intersection contains *exactly* one point, suppose that's not the case. This means we can find $x \ne y \in \bigcap_n E_n$. Now, for sufficiently large $n$, we have diam $E_n < d(x, y)$, contradicting $x, y \in E_n$. The contradiction negates the existence of $x \ne y \in \bigcap_n E_n$, so we're done. ■

**Solution for Problem 22, page 82.** We will follow the hint and construct a shrinking sequence of neighborhoods $E_n$ as suggested in the text of the problem.

The construction is recursive. In the first step, choose $E_1$ to be some neighborhood $E_1 = N_{\delta_1}(x_1)$ for some $x_1 \in G_1$ (this is possible because $G_1$ is open, so some neighborhood of an arbitrary point $x_1 \in G_1$ will be contained in it). Shrinking $\delta_1$ if necessary, we can ensure that

$$\overline{E_1} = \{y \in X :\ d(x_1, y) \le \delta_1\}$$

is contained in $G_1$.

For our second step, note that $G_2 \cap E_1$ is non-empty because $G_2$ is dense as well as open by Theorem 2.24 (c), so we can find some point $x_2 \in G_2 \cap E_1$ and a neighborhood $E_2 = N_{\delta_2}(x_2)$ contained in $G_2 \cap E_1$. Moreover, shrinking $\delta_2$ if necessary, we can ensure that $\delta_2 < \frac{\delta_1}{2}$ and that $\overline{E_2} \subset G_2 \cap E_1$.

Now continue as above. In step $k \ge 3$, assuming we've chosen sets $E_j$ for $1 \le j \le k-1$, select some point

$$x_k \in G_k \cap E_{k-1}$$

and a neighborhood $E_k = N_{\delta_k}(x_k)$ so that $\overline{E_k}$ is contained in $G_k \cap E_{k-1}$, and $\delta_k < \frac{\delta_{k-1}}{2}$.

This last inequality ensures that $\delta_k \to 0$ as $k \to \infty$. Since the diameter of $\overline{E_k} = \overline{N_{\delta_k}(x_k)}$ is at most $2\delta_k$, we get

$$\operatorname{diam} \overline{E_k} \to 0.$$

Applying Problem 21 to the sets $\overline{E_k}$ we get a point $x$ contained in their intersection, and since $\overline{E_k} \subset G_k$ we get $x \in \bigcap_k G_k$.

Moreover, you can do everything above inside an arbitrary non-empty open subset $U$ of $X$ to conclude that $\bigcap_k G_k$ contains points in $U$, and hence the intersection is dense in $X$ (as claimed in passing in the statement of the problem). ∎

**Solution for Problem 2, page 98.** We need to show that $f$ maps $\overline{E}$ into $\overline{f(E)}$, or equivalently, that

$$\overline{E} \subseteq f^{-1}\left(\overline{f(E)}\right). \tag{2}$$

Now, $\overline{f(E)}$ is closed in $Y$, so the right hand side of (2) is closed in $X$ by the corollary to Theorem 4.8. On the other hand we have $f(E) \subseteq \overline{f(E)}$, so

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\left(\overline{f(E)}\right).$$

In conclusion, the right hand side of (2) is a closed subset of $X$ which contains $E$. Since $\overline{E}$ is the *smallest* closed subset of $X$ containing $E$ (Theorem 2.27 (c)), (2) follows.

To show that the inclusion can be proper, consider the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x},$$

where both $(0,1)$ and $\mathbb{R}$ are equipped with the usual metric.

Take $E$ to be all of $(0,1)$, so that $E = \overline{E}$ and

$$f(E) = f(X) = (1,\infty) \subset \mathbb{R}.$$

The closure $\overline{f(E)}$ is then $[1,\infty)$, but the point $1 \in \overline{f(E)}$ is not in the image of $E = \overline{E}$. ∎

**Solution for Problem 3, page 98.** By definition, for $f : X \to \mathbb{R}$ the zero set $Z(f)$ of $f$ is the preimage $f^{-1}(\{0\})$ of the subset $\{0\} \subset \mathbb{R}$.

Since the singleton $\{0\} \subset \mathbb{R}$ is clearly closed (Example 2.21 (c)), its preimage through the continuous function $f$ is closed in $X$ by the corollary to Theorem 4.8. ∎

**Solution for Problem 4, page 98.** We first prove the claim that $f(E)$ is dense in $f(X)$. By the definition of density, this means that we want

$$f(X) \subseteq \overline{f(E)}, \tag{3}$$

where the closure on the right hand side is taken inside $Y$. Now, since $E$ is dense in $X$, we have $X = \overline{E}$, so the left hand side of (3). Hence, by Problem 2 we have

$$f(X) = f\left(\overline{E}\right) \subseteq \overline{f(E)}.$$

We now prove the second claim, that $f = g$ provided $f(p) = g(p)$ for $p \in E$.

Choose some arbitrary $q \in X$, with the goal of proving that $f(q) = g(q)$. Fix some $\varepsilon > 0$. Since $f$ and $g$ are continuous, there is some $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{2} \quad \text{and} \quad d(g(p), g(q)) < \frac{\varepsilon}{2} \quad \text{whenever} \quad d(p, q) < \delta. \tag{4}$$

Now, since $E$ is dense in $X$, we can find $p \in E$ with $d(p, q) < \delta$. Then, by (4) and the triangle inequality we have

$$d(f(q), g(q)) \leq d(f(q), f(p)) + d(f(p), g(p)) + d(g(p), g(q)).$$

The middle term on the right hand side is zero because $f(p) = g(p)$ by assumption (since $p \in E$), while the other two terms are both less than $\frac{\varepsilon}{2}$ by (4). All in all, we get $d(f(q), g(q)) < \varepsilon$. As this is true for *any* $\varepsilon > 0$, we have $f(q) = g(q)$ as desired. ∎

**E1.** Let $f : X \to Y$ be a map between metric spaces such that for all subsets $E \subseteq X$ we have

$$f\left(\overline{E}\right) \subseteq \overline{f(E)}.$$

Show that $f$ is continuous.

**Solution.** According to the corollary to Theorem 4.8, I have to show that for every closed subset $C \subseteq Y$ the preimage $f^{-1}(C)$ is closed.

Fix a closed subset $C \subseteq Y$, and set $E = f^{-1}(C) \subseteq X$. The task now is to prove that $E$ is closed, or in other words, that $\overline{E}$ is contained in $E$ (the other inclusion $E \subseteq \overline{E}$ being true by the definition of $\overline{E}$).

Now, by our definition of $E$, showing that

$$\overline{E} \subseteq E = f^{-1}(C)$$

simply means showing that $f\left(\overline{E}\right)$ is contained in $C$. Now, by assumption we have

$$f\left(\overline{E}\right) \subseteq \overline{f(E)}. \tag{5}$$

Since $f(E) \subseteq C$ and $C$ is closed, we further have

$$\overline{f(E)} \subseteq C$$

by Theorem 2.27 (c). This, together with (5) gives the desired conclusion $f\left(\overline{E}\right) \subseteq C$. ∎