Math 424 B / 574 B Due Monday, Nov 09
Autumn 2015

Solutions for Homework 5

Solution for Problem 20, page 82. We are starting out with a Cauchy sequence $(p_n)_n$ in a metric space (X, d) under the assumption that subsequence $(p_{n_\ell})_\ell$ converges to a point p. We then want to show that the original sequence $(p_n)_n$ converges to p.

Fix a positive real number $\delta > 0$. The Cauchy property for (p_n) ensures that there is some positive integer N such that

$$d(p_n, p_m) < \frac{\delta}{2},\ \forall n, m \geq N. \tag{1}$$

On the other hand, since $\lim_\ell p_{n_\ell} = p$, there is some positive integer M such that

$$d(p_{n_\ell}, p) < \frac{\delta}{2},\ \forall \ell \geq M. \tag{2}$$

Now set $L = \max\{n_M, N\}$, and fix some $k \geq M$ such that $n_k \geq L$. For $n \geq L$ we have

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p). \tag{3}$$

by the triangle inequality. Now, we are assuming that both n and n_k are greater than or equal to $L \geq N$, and hence the first term in the right hand side of (3) is smaller than $\frac{\delta}{2}$ by (1). On the other hand, $k \geq M$ so the second term of the right hand side is smaller than $\frac{\delta}{2}$ by (2).

All in all we get

$$d(p_n, p) < \delta,\ \forall n \geq L.$$

Since $\delta > 0$ was arbitrary, this proves that $p_n \to p$, as desired. ∎

E1. Let (X, d) be a metric space and $(x_n)_{n>0}$ a Cauchy sequence in X. Show that the set $\{x_n\}_{n>0}$ is bounded.

Solution. By the Cauchy property, there is some N such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. Now, for any positive integer m we have $d(x_N, x_m) < 1$ if $m \geq N$, or

$$d(x_N, x_m) \leq L = \max_{i=1}^{N-1} d(x_N, x_i)$$

if m is one of the numbers $1, \cdots, N-1$. Either way, the distance $d(x_N, x_m)$ is smaller than $1 + L$ and hence all terms x_m of the sequence are contained in the neighborhood $N_{1+L}(x_N)$. This is what it means to be bounded, so we are done. ∎

E2. Show that a subsequence of a Cauchy sequence is Cauchy.

Solution. Let $(x_n)_n$ be a Cauchy sequence in a metric space (X, d), and $y_k = x_{n_k}$ a subsequence. Fix a positive real number $\delta > 0$. The Cauchy property for (x_n) says that there is some positive integer N such that $d(x_n, x_m) < \delta$ whenever $n, m \geq N$.

Since the sequence $(n_k)_k$ of positive integers is strictly increasing, there is an M such that $n_k \geq N$ for all $k \geq M$. But then, for $k, \ell \geq M$, we have

$$d(y_k, y_\ell) = d(x_{n_k}, x_{n_\ell}) < \delta$$

because both n_k and n_ℓ are greater than or equal to M. Since $\delta > 0$ was arbitrary, we have verified the Cauchy property for the sequence $(y_k)_k$. ∎