Solutions for Homework 5

Solution for Problem 20, page 82. We are starting out with a Cauchy sequence \((p_n)_n\) in a metric space \((X,d)\) under the assumption that subsequence \((p_{n_\ell})_{\ell}\) converges to a point \(p\). We then want to show that the original sequence \((p_n)_n\) converges to \(p\).

Fix a positive real number \(\delta > 0\). The Cauchy property for \((p_n)_n\) ensures that there is some positive integer \(N\) such that
\[
d(p_n, p_m) < \frac{\delta}{2}, \quad \forall n, m \geq N. \tag{1}
\]
On the other hand, since \(\lim_{\ell} p_{n_\ell} = p\), there is some positive integer \(M\) such that
\[
d(p_{n_\ell}, p) < \frac{\delta}{2}, \quad \forall \ell \geq M. \tag{2}
\]
Now set \(L = \max\{n_M, N\}\), and fix some \(k \geq M\) such that \(n_k \geq L\). For \(n \geq L\) we have
\[
d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p). \tag{3}
\]
by the triangle inequality. Now, we are assuming that both \(n\) and \(n_k\) are greater than or equal to \(L \geq N\), and hence the first term in the right hand side of (3) is smaller than \(\frac{\delta}{2}\) by (1). On the other hand, \(k \geq M\) so the second term of the right hand side is smaller than \(\frac{\delta}{2}\) by (2).

All in all we get
\[
d(p_n, p) < \delta, \quad \forall n \geq L.
\]
Since \(\delta > 0\) was arbitrary, this proves that \(p_n \to p\), as desired. \(\blacksquare\)

\textbf{E1.} Let \((X,d)\) be a metric space and \((x_n)_{n>0}\) a Cauchy sequence in \(X\). Show that the set \(\{x_n\}_{n>0}\) is bounded.

\textbf{Solution.} By the Cauchy property, there is some \(N\) such that \(d(x_n, x_m) < 1\) for all \(n, m \geq N\). Now, for any positive integer \(m\) we have \(d(x_N, x_m) < 1\) if \(m \geq N\), or
\[
d(x_N, x_m) \leq L = \frac{N-1}{\max_{i=1}^N d(x_N, x_i)}
\]
if \(m\) is one of the numbers 1, \ldots, \(N-1\). Either way, the distance \(d(x_N, x_m)\) is smaller than \(1 + L\) and hence all terms \(x_m\) of the sequence are contained in the neighborhood \(N_{1+L}(x_N)\). This is what it means to be bounded, so we are done. \(\blacksquare\)

\textbf{E2.} Show that a subsequence of a Cauchy sequence is Cauchy.
Solution. Let \((x_n)_n\) be a Cauchy sequence in a metric space \((X,d)\), and \(y_k = x_{n_k}\) a subsequence. Fix a positive real number \(\delta > 0\). The Cauchy property for \((x_n)\) says that there is some positive integer \(N\) such that \(d(x_n, x_m) < \delta\) whenever \(n, m \geq N\).

Since the sequence \((n_k)_k\) of positive integers is strictly increasing, there is an \(M\) such that \(n_k \geq N\) for all \(k \geq M\). But then, for \(k, \ell \geq M\), we have

\[d(y_k, y_\ell) = d(x_{n_k}, x_{n_\ell}) < \delta\]

because both \(n_k\) and \(n_\ell\) are greater than or equal to \(M\). Since \(\delta > 0\) was arbitrary, we have verified the Cauchy property for the sequence \((y_k)_k\). \(\blacksquare\)