Math 424 B / 574 B Due Wednesday, Nov 04
Autumn 2015

Solutions for Homework 4

None of the problems are taken directly from the book, but the first two are variants of 19 and 20 respectively from the end of Chapter 2.

For the first problem, I need the following definition.

Definition 1. Let (X, d) be a metric space, and $A \subseteq E \subseteq X$ subsets. We say that A is *relatively closed in* E (or just *closed in* E) if it is a closed subset of E when I regard the latter as a metric space in its own right with respect to the metric d inherited from X. ♦

Remark 2. It can be shown that A is relatively closed in E if and only if it is of the form $E \cap F$, where $F \subseteq X$ is closed. You can use this below. ♦

1. Let A and B be two disjoint and non-empty subsets of a metric space X, $E \subseteq X$ another subset, and assume $E = A \cup B$. Prove that the following statements are equivalent.

- A and B are separated;
- A and B are both relatively closed in E;
- A and B are both relatively open in E.

Solution. We will cycle through the three conditions, proving three implications that go around in a circle. I will use the following characterization of relatively open / closed sets: A is relatively closed (open) in E if and only if it is of the form $E \cap S$, where S is a closed (respectively open) subset of X.

This is Theorem 2.30 for the open case, whereas the proof for the closed case is very similar (see also Remark 2).

First • implies second • According to Definition 2.45, separatedness means

$$A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$

Now, we have $B \subseteq \overline{B}$, so $B \subseteq E \cap \overline{B}$. Because $\overline{B}$ does not intersect A though, we also have

$$E \cap \overline{B} \subseteq E \setminus A = B.$$

All in all we get $B = E \cap \overline{B}$. This is the intersection of E with a closed set, so it's relatively closed. Similarly, A is relatively closed in E (simply interchange the roles of A and B).

Second • implies third • Since B is relatively closed, it is of the form $E \cap F$ for some closed subset $F \subseteq X$. But then we have

$$A = E \setminus B = E \setminus (E \cap F) = E \cap F^c.$$

The complement F^c is open because F is closed (Theorem 2.23), so that A is relatively open in E by Theorem 2.30.

Again, interchanging the roles of A and B in the argument above shows that B is also relatively open.

Third • implies first • We are now assuming that A is relatively open in E. Fix some arbitrary point $a \in A$. Relative openness in E means that for some $r > 0$ the intersection $N_r(a) \cap E$ is contained in $A = E \setminus B$. But this then implies that $N_r(a)$ cannot contain any points from B, and hence $a \notin \overline{B}$.

Since $a \in A$ was arbitrary, the above paragraph shows that $A \cap \overline{B} = \emptyset$. Once more, interchanging the roles of A and B proves the other condition $\overline{A} \cap B = \emptyset$. ■

What you've just proven is the following alternate definition of connectedness; this is the one that's more commonly used in practice.

Definition 3. A metric space is connected if it cannot be written as the union of two disjoint, non-empty closed (or open) subsets. ♦

2. Suppose A is a connected subset of a metric space.

(a) Show that the closure $\overline{A}$ is also connected.

(b) Show by example that the interior A° need not be connected.

Hint for part (b): Try to come up with two subsets of $\mathbb{R}^2$ that just barely touch each other and such that both have non-empty interiors.

Solution. (a) We will prove this by contradiction. Suppose $\overline{A}$ is not connected. Then, by Definition 3, I can write it as a disjoint union

$$\overline{A} = B \cup C$$

of two non-empty relatively closed subsets B and C. Relatively closed subsets of $\overline{A}$ are of the form $\overline{A} \cap E$ for some closed $E \subseteq X$ (this is the analogue of Theorem 2.30, with 'closed' instead of 'open'). $\overline{A}$ itself is closed and intersections of closed sets are closed (by part (b) of Theorem 2.24), so that both B and C must be closed.

Now, A is the disjoint union of the two relatively closed subsets $A \cap B$ and $A \cap C$. Since A is connected, Definition 3 ensures that one of these two sets must be empty. Without loss of generality, let's say $A \cap C = \emptyset$. But this, together with

$$A = (A \cap B) \cup (A \cap C),$$

implies $A \subseteq B$. Since B is closed, part (c) of Theorem 2.27 shows that $\overline{A} \subseteq B$; this contradicts our assumption that $C = \overline{A} \setminus B$ is non-empty.

(b) My ambient metric space will be $\mathbb{R}^2$ with the usual metric coming from the identification $\mathbb{R}^2 \cong \mathbb{C}$. The set A will be

$$\{p \in \mathbb{R}^2 : \ d(p, 0) \leq 1 \text{ or } d(p, 2) \leq 1\}.$$

In other words, A is the union of the two closed disks of radius 1 centered at the points 0 and 2. These disks just barely touch at 1, as the hint suggests. Because each of the disks is connected and their intersection is non-empty (as they intersect at 1), their union A is connected by Problem 3 below.

The interior of A is the union

$$\{p \in \mathbb{R}^2 : \ d(p,0) < 1 \text{ or } d(p,2) < 1\}.$$

of the two *open* disks centered at the same two points 0 and 2. These disks do not intersect, so the interior is the disjoint union of two non-empty open subsets. This violates Definition 3 above, so that A° is not connected. ■

3. Let $\{A_i\}_{i\in I}$ be a family of connected subsets of a metric space X (where I is just some set of indices; it need not be finite or countable).

Show that if the intersection $\bigcap_i A_i$ is non-empty, then the union $\bigcup_i A_i$ is connected.

Solution. Denote the union $\bigcup_i A_i$ by A. According to Definition 3 above, we have to show that if A is the union of two disjoint relatively open subsets $A\cap U$ and $A\cap V$ (where U and V are open subsets of X), then one of these two sets must be empty.

Now, our assumption that

$$A = (A\cap U)\cup(A\cap V)$$

implies that we similarly have

$$A_i = (A_i\cap U)\cup(A_i\cap V)$$

for each i. The relatively open subsets $A_i\cap U$ and $A_i\cap V$ of A_i are disjoint (because the larger sets $A\cap U$ and $A\cap V$ are), so the connectedness of A_i ensures, by Definition 3, that one of $A_i\cap U$ and $A_i\cap V$ must be empty.

So far, we've determined that for each i, either $A_i\cap U$ or $A_i\cap V$ is empty. Which one is empty might depend on i though. To get around this issue, suppose $A_j\cap U=\emptyset$ and $A_k\cap V=\emptyset$ for some $j\ne k\in I$. We then have

$$\left(\bigcap_i A_i\cap U\right)\cup\left(\bigcap_i A_i\cap V\right)\subseteq (A_j\cap U)\cup(A_k\cap V)=\emptyset.$$

The left hand side of this expression, however, is all of $\bigcap_i A_i$ (because U and V cover all of A by assumption), and our hypothesis says that this intersection is non-empty. The contradiction means that we cannot have $A_j\cap U$ and $A_k\cap V$ empty for $j\ne k$, and the conclusion is that either $A_i\cap U$ is empty for *all* $i\in I$, or similarly, $A_i\cap V$ is empty for all i.

But then we have either

$$A\cap U=\bigcup_i(A_i\cap U)=\emptyset$$

or similarly $A\cap V=\emptyset$. This was our goal (see the first paragraph of the solution), so we are done. ■

In particular, this problem shows that the union of all connected subsets of X containing a given point $x\in X$ is again connected. In other words, there's a *largest* connected subset of X containing x. This validates the following definition.

Definition 4. For $x \in X$ the *connected component* of x is the largest connected subset of X containing x. ♦

Note that by part (a) of Problem 2 the connected component of any point of X is a closed subset of X.

For the next problem I'll need the following notion.

Definition 5. Let $A \subseteq X$ be a subset of a metric space X. The *boundary* ∂A of A is the intersection

$$\overline{A} \cap \overline{A^c}.$$

Equivalently, it is the set of points in X whose neighborhoods contain points from both A and the complement $A^c = X \setminus A$. ♦

4. Show that a metric space X is connected if and only if for every non-empty proper subset $A \subset X$ the boundary ∂A is non-empty (*proper* means A is *not* all of X).

Solution. (⇒) We prove this implication by contradiction. Suppose that X is connected, and let $A \subset X$ be a proper non-empty subset with $\partial A = \emptyset$. A is non-empty, so the closed set $\overline{A} \supseteq A$ is also non-empty. Similarly, because A is proper, the complement A^c is non-empty and so is the closure $\overline{A^c}$. We have

$$X = A \cup A^c \subseteq \overline{A} \cup \overline{A^c} \subseteq X,$$

so X is the union of the two closed sets $\overline{A}$ and $\overline{A^c}$. We are furthermore assuming that these sets do not intersect (this is what $\partial A = \emptyset$ means), so we have exhibited X as a disjoint union of two non-empty closed subsets. This violates Definition 3, contradicting connectedness.

(⇐) I will prove the contrapositive of this implication. Suppose X is not connected. Then, according to Definition 3, I can write it as a disjoint union $A \cup B$ of non-empty closed subsets. But then $B = A^c$ is closed and hence equal to its own closure (and similarly $A = \overline{A}$), and we have

$$\partial A = \overline{A} \cap \overline{A^c} = A \cap A^c = \emptyset.$$

Since A was proper (because $B = A^c \neq \emptyset$) and non-empty, this negates the property in the statement of the problem (that non-empty proper subsets of X have non-empty boundaries). This concludes the proof. ■

5. Define

$$x_n = \left(1 + \frac{1}{n}\right) \sin \frac{n\pi}{2}.$$

Find

(a) All the limits of convergent subsequences of the *sequence* (x_n);
(b) All the limit points of the *set* $\{x_n : \; n = 1, 2, \cdots\}$.

Solution. **(a)** The behavior of x_n changes depending on the remainder of n upon division by 4.

First, if n is even, then $\sin\dfrac{n\pi}{2}$ is zero, and so $x_n = 0$ for those values of n. In conclusion, we have a subsequence

$$y_k = x_{2k},\ k = 1, 2, \cdots$$

that is constantly zero and so converges to 0.

Secondly, if n is of the form $4k+1$ for $k \geq 0$ (that is, $n = 1, 5, 9, \cdots$) then $\sin\dfrac{n\pi}{2}$ equals 1 and we have $x_n = 1 + \frac{1}{n}$. We thus have a subsequence

$$z_k = x_{4k+1} = 1 + \frac{1}{4k+1},\ k = 0, 1, \cdots$$

which converges to 1 as $k \to \infty$.

Finally, if n is of the form $4k+3$ (i.e. $n = 3, 7, 11, \cdots$) then $\sin\dfrac{n\pi}{2}$ equals -1 and we have $x_n = -\left(1 + \frac{1}{n}\right)$. We thus have a subsequence

$$w_k = x_{4k+3} = -\left(1 + \frac{1}{4k+3}\right),\ k = 0, 1, \cdots$$

which converges to -1 as $k \to \infty$.

So the limits the problem asks for are 0 and ± 1.

(b) As observed above in part (a), the set $\{x_n\}$ consists of the numbers $1 + \frac{1}{4k+1}$ for $k \geq 0$, the numbers $-\left(1 + \frac{1}{4k+3}\right)$ for $k \geq 0$, and 0 (the common value of all x_n when n is even).

The only limit point of the first set

$$\left\{1 + \frac{1}{4k+1}\ :\ k = 0, 1, \cdots\right\}$$

is 1, and similarly, the only limit point of

$$\left\{-\left(1 + \frac{1}{4k+3}\right)\ :\ k = 0, 1, \cdots\right\}$$

is -1. The point 0 is isolated (i.e. it has a neighborhood, say $(-1, 1)$, that contains no other points of $\{x_n\}$), and so contributes nothing to the limit points.

In conclusion, the only limit points are ± 1. ∎

6. Let (X, d) be a metric space.

(a) Show that for all $x, y, p, q \in X$ we have

$$|d(x,y) - d(p,q)| \leq d(x,p) + d(y,q).$$

(b) Conclude that if $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.

Solution. **(a)** Applying the triangle inequality twice, we get

$$d(x,y) \leq d(x,p) + d(p,y) \leq d(x,p) + d(p,q) + d(y,q).$$

Moving $d(p,q)$ to the left, this implies

$$d(x,y) - d(p,q) \leq d(x,p) + d(y,q). \tag{1}$$

Now interchange the roles of x and p, and also the roles of y and q. Applying the triangle inequality twice, as above, we get

$$d(p,q) \leq d(x,p) + d(x,q) \leq d(x,p) + d(x,y) + d(y,q).$$

Moving $d(x,y)$ to the left yields

$$d(p,q) - d(x,y) \leq d(x,p) + d(y,q). \tag{2}$$

(1) and (2) together show that the absolute value of the number $d(x,y) - d(p,q)$ is dominated by the right hand side of both equations, as desired.

(b) Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of convergence, there is some N such that $d(x_n, x)$ and $d(y_n, y)$ are both less than $\frac{\varepsilon}{2}$ for $n \geq N$. But then, applying part (a) with $p = x_n$ and $q = y_n$, we get

$$|d(x_n, y_n) - d(x,y)| \leq d(x, x_n) + d(y, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for all $n \geq N$. This means that the sequence $(d(x_n, y_n))_n$ does indeed converge to the number $d(x,y)$. ■