Math 424 B / 574 B Due Wednesday, Nov 04
Autumn 2015

Homework 4

None of the problems are taken directly from the book, but the first two are variants of 19 and 20 respectively from the end of Chapter 2.

For the first problem, I need the following definition.

Definition 1. Let (X, d) be a metric space, and $A \subseteq E \subseteq X$ subsets. We say that A is *relatively closed in* E (or just *closed in* E) if it is a closed subset of E when I regard the latter as a metric space in its own right with respect to the metric d inherited from X. ♦

Remark 2. It can be shown that A is relatively closed in E if and only if it is of the form $E \cap F$, where $F \subseteq X$ is closed. You can use this below. ♦

1. Let A and B be two disjoint and non-empty subsets of a metric space X, $E \subseteq X$ another subset, and assume $E = A \cup B$. Prove that the following statements are equivalent.

- A and B are separated;
- A and B are both relatively closed in E;
- A and B are both relatively open in E.

What you've just proven is the following alternate definition of connectedness; this is the one that's more commonly used in practice.

Definition 3. A metric space is connected if it cannot be written as the union of two disjoint, non-empty closed (or open) subsets. ♦

2. Suppose A is a connected subset of a metric space.

(a) Show that the closure $\overline{A}$ is also connected.
(b) Show by example that the interior A° need not be connected.

Hint for part (b): Try to come up with two subsets of $\mathbb{R}^2$ *that just barely touch each other and such that both have non-empty interiors.*

3. Let $\{A_i\}_{i \in I}$ be a family of connected subsets of a metric space X (where I is just some set of indices; it need not be finite or countable).

Show that if the intersection $\bigcap_i A_i$ is non-empty, then the union $\bigcup_i A_i$ is connected.

In particular, this problem shows that the union of all connected subsets of X containing a given point $x \in X$ is again connected. In other words, there's a *largest* connected subset of X containing x. This validates the following definition.

Definition 4. For $x \in X$ the *connected component* of x is the largest connected subset of X containing x. ♦

Note that by part (a) of Problem 2 the connected component of any point of X is a closed subset of X.

For the next problem I'll need the following notion.

Definition 5. Let $A \subseteq X$ be a subset of a metric space X. The *boundary* ∂A of A is the intersection

$$\overline{A} \cap \overline{A^c}.$$

Equivalently, it is the set of points in X whose neighborhoods contain points from both A and the complement $A^c = X \setminus A$. ♦

4. Show that a metric space X is connected if and only if for every non-empty proper subset $A \subset X$ the boundary ∂A is non-empty (*proper* means A is *not* all of X).

5. Define

$$x_n = \left(1 + \frac{1}{n}\right) \sin \frac{n\pi}{2}.$$

Find

(a) All the limits of convergent subsequences of the *sequence* (x_n);
(b) All the limit points of the *set* $\{x_n : \ n = 1, 2, \cdots\}$.

6. Let (X, d) be a metric space.

(a) Show that for all $x, y, p, q \in X$ we have

$$|d(x, y) - d(p, q)| \leq d(x, p) + d(y, q).$$

(b) Conclude that if $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.