Math 424 B / 574 B Due Wednesday, Oct 21
Autumn 2015

Solutions for Homework 3

Solution for Problem 6, page 43. Let's break it up according to the different tasks.

Part (1): E' is closed. I have to show that E' contains its limit points.

Let $x \in E''$ be a limit point of E' (the double-prime notation means $(E')'$, i.e. the set of limit points of the set of limit points of E; the repetition is not a typo!). Let $N_r(x)$ be an arbitrary neighborhood of x. By the definition of a limit point, $N_r(x)$ contains some point $y \in E'$ that is different from x.

Now choose $s > 0$ so small that $N_s(y) \subset N_r(x)$ and $x \notin N_s(y)$. For instance, you could take

$$s < \min(d(x, y), r - d(x, y)).$$

The inequality $s < d(x, y)$ ensures that $x \notin N_s(y)$, while the inequality $s < r - d(x, y)$ means that for every $y' \in N_s(y)$ we have

$$d(x, y') \leq d(x, y) + d(y, y') < d(x, y) + s < r,$$

so that $y' \in N_r(x)$. Or in other words, $N_s(y) \subset N_r(x)$, as desired.

Now, y itself was in E' (I chose it so), which means that $N_s(y)$ contains some point $z \in E$ different from y. But now we have

$$E \ni z \in N_s(y) \subset N_r(x),$$

and $z \neq x$ because z is in $N_s(y)$, which doesn't contain x.

In conclusion, we have found, in the arbitrary neighborhood $N_r(x)$ of x, a point $z \in E$ that is different from x. This means that $x \in E'$. But since x was an arbitrary point of E'', we get the desired inclusion $E'' \subseteq E'$.

Part (2): E and $\overline{E}$ have the same limit points. It is clear from the definition of a limit point that if $E \subseteq F$, then $E' \subseteq F'$. In other words, the operation $E \mapsto E'$ respects inclusions.

$\overline{E} = E \cup E'$ contains E, so by the above observation we have $E' \subseteq \left(\overline{E}\right)'$.

We now need the opposite inclusion $\left(\overline{E}\right)' \subseteq E'$. Let $x \in \left(\overline{E}\right)'$ be an arbitrary point, and let $N_r(x)$ be an arbitrary neighborhood for it. I have to find a point $z \in E$, different from x, in $N_r(x)$.

Since x is a limit point of $\overline{E}$, I can find some element $y \in \overline{E} = E \cup E'$ in $N_r(x)$ that is distinct from x. We now have $y \in E$ or $y \in E'$.

If $y \in E$, just take $z = y$.

If $y \in E'$, find $z \in E$ using the same argument as in Part (1).

This concludes the proof of Part (2).

Part (3): Do E and E' always have the same limit points? No: just take

$$E = \left\{ \frac{1}{n} :\ n > 0 \right\}.$$

As you've seen before (in Example 2.21 (e) and on previous homework) the set E' of limit points here is just $\{0\}$. But then E' has *no* limit points of its own (finite sets have no limit points, for example by Theorem 2.20). So we have

$$E'' = \emptyset \neq \{0\} = E',$$

and we have our example. ∎

Solution for Problem 7, page 43. Let me start with a general inclusion $B \supseteq A$ of subsets of a metric space. Then, I claim I get the analogous inclusion $\overline{B} \supseteq \overline{A}$.

To see this, note first that we have

$$\overline{B} \supseteq B \supseteq A,$$

so that $\overline{B}$ is a closed set containing A. But according to Theorem 2.27 the closure $\overline{A}$ is the *smallest* closed subset containing A, so I get my desired inclusion

$$\overline{B} \supseteq \overline{A}.$$

Part (b). Now let's specialize to the setup of the problem. Since all A_i are contained in their union B, I get $\overline{B} \supseteq \overline{A}_i$ for all i and hence the inclusion

$$\overline{B} \supseteq \bigcup_i \overline{A}_i.$$

from part (b).

Part (a). The same argument as in Part (b) above proves the inclusion

$$\overline{B}_n \supseteq \bigcup_{i \le n} \overline{A}_i \tag{1}$$

This time around though we want the opposite inclusion as well.

To get that, note first that we have

$$B_n = \bigcup_{i \le n} A_i \subseteq \bigcup_{i \le n} \overline{A}_i.$$

The rightmost set in this display is a finite union of closed sets, and hence closed by part (d) of Theorem 2.24. But again by Theorem 2.27, $\overline{B}_n$ is the *smallest* closed set containing B_n, and hence indeed

$$\overline{B}_n \subseteq \bigcup_{i \le n} \overline{A}_i.$$

Together with the opposite inclusion (1) this proves the equality required in part (a).

A counterexample. Finally, we have to give an example where the inclusion in part (b) is strict. For that, just take for instance $A_i = \left\{\frac{1}{i}\right\}$ for $i > 0$. Each A_i is closed, so

$$\bigcup_i \overline{A}_i = \bigcup_i A_i = \left\{\frac{1}{i} :\ i > 0\right\}. \tag{2}$$

On the other hand, 0, which does not belong to (2), is a limit point of the union

$$B = \bigcup_i A_i.$$

This means that it does $\overline{B}$, so the latter set is strictly larger than (2). ■

Solution for Problem 9, page 43. We do these in one by one.

(a) Running through the definitions of interior and openness, I have to show that if $x \in E^\circ$, then an entire neighborhood $N_r(x)$ of x consists of interior points of E.

Let us note first that if a set A is open, then by definition $A^\circ = A$. Applying this to the set $A = N_r(x)$ (which is open by Theorem 2.19), we get

$$N_r(x)^\circ = N_r(x). \tag{3}$$

A second observation is that if $x \in A^\circ$ and $A \subseteq B$, then x is also an interior point of B. This means that $A^\circ \subseteq B^\circ$, i.e. the operation $A \mapsto A^\circ$ preserves inclusions. Now apply this to $A = N_r(x)$ and $B = E$. Using (3) we get

$$N_r(x) = N_r(x)^\circ \subseteq E^\circ.$$

This is exactly what we wanted: some neighborhood of x is contained in E°.

(b) We already noted in the proof of part (a) above that if E is open then E° as an immediate consequence of the definitions.

On the other hand, since E° is open by part (a), $E = E^\circ$ implies that E is open. So the two conditions 'E is open' and '$E = E^\circ$' are indeed equivalent.

(c) Let $G \subseteq E$ be an open set.

We noted above, in the proof of part (a), that the inclusion $G \subseteq E$ implies $G^\circ \subseteq E^\circ$. On the other hand, the openness of G also implies (as noted in the same proof) that $G^\circ = G$. So all in all we get

$$G = G^\circ \subseteq E^\circ.$$

(d) An element x of the ambient metric space X is in the complement $(E^\circ)^c$ if and only if it is *not* an interior point of E. Running through the definition of interior points, this means that for every $r > 0$ the neighborhood $N_r(x)$ contains elements of E^c.

This last condition is equivalent to saying that either $x \in E^c$, or if not, every neighborhood $N_r(x)$ contains elements of E^c that are necessarily different from x (because we are in the case $x \notin E^c$). In other words, $x \in E^c$ or x is a limit point of E^c.

In conclusion,

$$x \in (E^\circ)^c \Leftrightarrow x \in E^c \cup (E^c)' = \overline{E^c}.$$

This is what we were asked to prove.

(e) An example where E has empty interior but $\overline{E}$ does not would show that E and $\overline{E}$ do not always have the same interior.

Take E to be, say, the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers, sitting inside the ambient metric space $\mathbb{R}$ (with the usual metric).

First, the set E is dense in $\mathbb{R}$, so that $\overline{E} = \mathbb{R}$. The density claim amounts to saying that every non-empty open interval in $\mathbb{R}$ contains irrational numbers; of course it does, since we know that every such interval is uncountable (well, we know this for $\mathbb{R}$, but the same proof works), so it can't possibly be that the interval consists of rational numbers alone.

In conclusion, $\overline{E} = \mathbb{R}$ which is open (as a subset of itself), so that the interior of $\overline{E}$ is all of $\mathbb{R}$.

On the other hand, the interior of E itself is empty: indeed, by part (b) of Theorem 1.20 every non-empty open interval in $\mathbb{R}$ contains rational numbers, so it can't be included in $E = \mathbb{R} \setminus \mathbb{Q}$.

We now have

$$E^\circ = \emptyset \neq \mathbb{R} = \left(\overline{E}\right)^\circ.$$

(f) That E and E° do not always have the same closures will certainly follow if I come up with an example where E is non-empty but E° is (because then the closure of E° would also be empty, while that of E wouldn't).

One simple example of this is, say, $E = \{0\}$ in $\mathbb{R}$. It's certainly non-empty, but its interior is empty (otherwise it would have to be E itself, but E is not open). ■

Solution for Problem 12, page 44. I need to show that for any open cover $K \subset \bigcup_\alpha G_\alpha$ I can extract a finite subcover. In other words, I can cover

$$K = \left\{\frac{1}{n} : \ n > 0\right\} \cup \{0\}$$

with only finitely many of the G_α.

The element 0 is in K and the G_α collectively cover K, so for sure *some* G_{α_0} contains 0. But G_{α_0} is open, so by the definition of openness there is a neighborhood $N_r(0) = (-r, r) \subset G_{\alpha_0}$ of 0.

Now, for sufficiently large n (for instance for $n > \frac{1}{r}$) the elements $\frac{1}{n}$ of K are all inside $N_r(0)$ and hence inside G_{α_0}.

In other words, most of the elements of K (i.e. $\frac{1}{n}$ for sufficiently large n along with 0) are contained in this single open set G_{α_0}. There are only finitely many $\frac{1}{n}$ left over that are maybe not in G_{α_0}, and we can cover those with finitely many other G_αs.

So all in all, we have used only finitely many G_α to cover K. ■

Solution for Problem 15, page 44. To give closed but unbounded counterexamples to the corollary to Theorem 2.36, take $K_n = [n, \infty)$. Clearly, this is an unbounded set. It's closed because it is the complement of the set $(-\infty, n)$, and this latter set is open by part (a) of Theorem 2.24 because it is the union of the neighborhoods (m, n) as m ranges over the integers going off to $-\infty$.

We also have $K_n \supset K_{n+1}$, as the corollary requests, but the intersection $\bigcap_n K_n$ is empty. Indeed, it consists of real numbers y such that $y > n$ for all positive integers n. This contradicts party (a) of Theorem 1.20 (applied to $x = 1$).

On the other hand, for a counterexample to the corollary consisting of sets that are bounded but *not* closed, take $K_n = \left(0, \frac{1}{n}\right)$.

Clearly, the K_ns are bounded and nested, in the sense that $K_n \supset K_{n+1}$, just as the corollary demands. Their intersection, however, consists of positive real numbers x that are smaller than $\frac{1}{n}$ for all n. This is the same as

$$0 < nx < 1, \ \forall n > 0,$$

which for example contradicts part (a) of Theorem 1.20 (with $y = 1$). So there are no such numbers x, and the intersection is empty.

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Solution for Problem 23, page 45. We will follow the hint in the book.

Let X be a separable metric space, and $\{x_n : \ n = 1, 2, \cdots\}$ a dense countable subset of X. Now, for each positive rational number $r > 0$ and each positive integer n, define

$$G_{n,r} = N_r(x_n).$$

In other words, this is the neighborhood of rational radius r centered at x_n. These are going to be our sets G_α requested by the problem: the indices α are the pairs (n, r) consisting of a positive integer n and a positive rational r. My task is now to show that they make up a countable base.

First, note that $\{G_{n,r}\}$ is indeed a countable family of open sets, because the set of pairs (n, r) as above is a(n infinite) subset of the set of all pairs of rational numbers, which is countable for instance by Theorem 2.13.

We are left having to show that the sets $G_{n,r}$ form a base, as defined in the statement of the problem. In other words, I have to show that for every $x \in X$ and every open set $G \ni x$ I can find some (n, r) such that

$$x \in G_{n,r} \subset G. \tag{4}$$

Now, G is open and it contains x, so by the definition of openness there is some neighborhood $N_s(x)$ of x that's contained in G.

Now let $0 < r < \frac{s}{2}$ be a positive rational number (one exists by part (b) of Theorem 1.20). Since $\{x_n\}$ is dense in x, there are elements of that set as close to x as I want. In particular, there is some n such that $d(x, x_n) < r$.

I now claim that $G_{n,r} = N_r(x_n)$ for the n and r from the previous paragraph satisfies (4). I have to show two things:

First, $x \in G_{n,r}$ because the distance from x to the center x_n of the neighborhood $G_{n,r}$ is less than r (by my choice of x_n).

Next, we want to prove that $G_{n,r}$ is contained in G. We will show that it is in fact contained in $N_s(x) \subset G$.

To see this, let $y \in G_{n,r} = N_r(x_n)$. This means that $d(y, x_n) < r$, and hence

$$d(y, x) \leq d(y, x_n) + d(x, x_n) < 2r, \tag{5}$$

where the first inequality is the triangle inequality for the distance function d and the second inequality is due to the fact that both summands are smaller than r.

On the other hand, we chose r so that it is less than $\frac{s}{2}$, and hence the rightmost term of (5) is less than s. But this means that an arbitrary $y \in G_{n,r}$ is contained in $N_s(x)$, as desired. ■