Math 424 B / 574 B Due Wednesday, Oct 14
Autumn 2015

Solutions for Homework 2

Solution for Problem 2, page 43. Clearly, the set of algebraic numbers is not finite (because it contains the integers, for example), so proving that it's countable is the same as proving that it is at most countable.

I will say that an algebraic number x has *degree* $\le n$ if there is a polynomial equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

with integers a_i that are not all zero.

By the definition of being algebraic, all algebraic numbers are of degree $\le n$ for *some* n. This means that if I denote the set of algebraic numbers of degree $\le n$ by A_n, then the set A of all algebraic numbers is the union of all A_n.

Since $A = \bigcup_n A_n$, the corollary to Theorem 2.12 from your book ensures that A is at most countable so long as each A_n is at most countable. This is now our goal for the rest of the proof: showing that for a fixed but arbitrary n, the set A_n is at most countable.

For each choice of $(n+1)$-tuple

$$\mathbf{a} = (a_0, \cdots a_n)$$

of integers that are not all zero, the equation (1) has only finitely many solutions.

Now, denote by $A_{\mathbf{a}}$ this finite set of solutions associated to the tuple $\mathbf{a}$. We have

$$A_n = \bigcup_{\mathbf{a}} A_{\mathbf{a}}, \tag{2}$$

where $\mathbf{a}$ ranges over all choices of $(n+1)$-tuples of integers (that are not all zero). This set of tuples is countable by Theorem 2.13, so (2) exhibits A_n as a union of countably many sets $A_{\mathbf{a}}$ that are at most countable (in fact finite). By the corollary of Theorem 2.12 again we find that A_n is at most countable. ∎

Solution for Problem 5, page 43. Example 2.21 (e) in the textbook is a bounded set (namely $\left\{\frac{1}{n} :\ n \in \mathbb{Z}_{>0}\right\}$) with exactly one limit point, 0.

If you want *three* limit points, it might be a good idea to somehow bring together three copies of that set. So let's shift it around a bit to get more copies of it; my example will be

$$A = \left\{\frac{1}{n}\right\} \cup \left\{1 + \frac{1}{n}\right\} \cup \left\{2 + \frac{1}{n}\right\},$$

where in all cases n ranges over the positive integers.

Since the set consists of the members of three sequences that converge to 0, 1 and 2 respectively, these three are going to be the limit points of A. Let's unpack that a bit.

First, my set A is bounded, for example because all of its members are bigger than 0 and no greater than 3.

Secondly, let's work out the limit points. Note that 0, 1 and 2 are indeed limit points: for 0 say, every neighborhood (which here just means open interval $(-r, r)$ centered at 0) contains elements of the set

$$\left\{\frac{1}{n} : \ n > 0\right\} \subset A.$$

This means that 0 meets the criterion for being a limit point (see part (b) of Definition 2.18). The argument is the same for the points 1 and 2, using the sets

$$\left\{1 + \frac{1}{n} : \ n > 0\right\} \subset A \quad \text{and} \quad \left\{2 + \frac{1}{n} : \ n > 0\right\} \subset A$$

respectively.

Finally, we have to check that there are no other limit points apart from 0, 1 and 2. Let $x \in \mathbb{R}$ be some number different from all three. This means I can find some small positive real numbers r and s such that

$$(-s, s) \cap (x - r, x + r) = (1 - s, 1 + s) \cap (x - r, x + r) = (2 - s, 2 + s) \cap (x - r, x + r) = \emptyset.$$

In other words, the neighborhoods $N_s(0)$, $N_s(1)$ and $N_s(2)$ do not intersect the neighborhood $N_r(x)$ (see the notation introduced in Definition 2.18).

If n is large enough, we have

$$\frac{1}{n} \in N_s(0), \ 1 + \frac{1}{n} \in N_s(1), \ \text{and } 2 + \frac{1}{n} \in N_s(2).$$

Since these neighborhoods do not intersect $N_r(x)$, only finitely many elements of A can fall inside $N_r(x)$. Shrinking this neighborhood further if necessary by choosing a smaller $r' < r$, I can find an interval $(x - r', x + r')$ that contains *no* elements of A apart maybe for x itself, if it happens to belong to A (which it may or may not). This means, according to part (b) of Definition 2.18, that x is not a limit point. ■

E1. Prove that for any two sets A and B we have

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A). \tag{3}$$

Solution. By the definition of set difference, an element belongs to the left hand side of (3) if and only if it is in $A \cup B$ but not in $A \cap B$. This is the same as saying that is in one of A and B but not in both.

Finally, this is equivalent to it being in A but not in B, or in B but not in A; that's exactly the right hand side of (3), and we're done. ■

E2. Let $f : X \to Y$ be a function between two sets. Show that for subsets $A, B \subseteq Y$ we have

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

and

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

Solution. Let us prove the first equation. An element $x \in X$ is in the inverse image $f^{-1}(A \cup B)$ if and only if $f(x) \in A \cup B$. This is in turn equivalent to

$$(f(x) \in A \quad \text{or} \quad f(x) \in B) \Leftrightarrow \left(x \in f^{-1}(A) \quad \text{or} \quad x \in f^{-1}(B)\right).$$

In turn, this last condition is equivalent to $x \in f^{-1}(A) \cup f^{-1}(B)$.

The second equation is proven similarly; just substitute the symbol '$\cap$' for '$\cup$' and the word 'and' for 'or'. ■

E3. Let $f : X \to Y$ be a map from X to Y

(a) Show that for subsets $C, D \subseteq X$ we have $f(C \cup D) = f(C) \cup f(D)$.
(b) For C and D as before, show that we also have $f(C \cap D) \subseteq f(C) \cap f(D)$.
(c) Take $X = Y = \mathbb{R}$ and $f(x) = x^2$. Find two subsets C and D of $\mathbb{R}$ such that $f(C \cap D) \neq f(C) \cap f(D)$.
(d) Going back to the general case of an arbitrary map $f : X \to Y$, show that the statement

$$f(C \cap D) = f(C) \cap f(D) \text{ for all possible choices of } C, D \subseteq X \tag{4}$$

is equivalent to f being one-to-one.

Solution. **(a)** An element $y \in Y$ is in $f(C \cup D)$ if and only if there is some $x \in C \cup D$ such that $f(x) = y$. This is equivalent to

$$\exists x \in C : \ f(x) = y \quad \text{or} \quad \exists x \in D : \ f(x) = y.$$

Finally, this is the same as

$$y \in f(C) \quad \text{or} \quad y \in f(D),$$

or in other words $y \in f(C) \cup f(D)$.

(b) Let $y \in f(C \cap D)$. This means that there is some $x \in C \cap D$ such that $f(x) = y$. But then, since $x \in C$, we have $y \in f(C)$; similarly for D, so $y \in f(C) \cap f(D)$.

(c) Take C and D to be the singletons $\{1\}$ and $\{-1\}$ respectively. Then $C \cap D = \emptyset$ so $f(C \cap D) = \emptyset$, but

$$f(C) \cap f(D) = f(C) = f(D) = \{1\}.$$

(d) Suppose first that f is one-to-one. We already know from part (b) above that $f(C \cap D)$ is contained in $f(C) \cap f(D)$, so we only have to prove the other inclusion.

Let $y \in f(C) \cap f(D)$. This means I can find $c \in C$ and $d \in D$ with $f(c) = f(d) = y$. But the property of being one-to-one ensures that $c \in d$, and this element then belongs to both C and D. In other words, we have found an element in $C \cap D$ whose image through f is y. This means exactly that $y \in f(C \cap D)$, concluding the proof of the inclusion

$$f(C) \cap f(D) \subseteq f(C \cap D).$$

Conversely, assume condition (4) holds. We will prove that f is one-to-one by contradiction.

Suppose f is *not* one-to-one. This means we can find $c \neq d \in X$ with $f(c) = f(d) = y \in Y$. Now take $C = \{c\}$ and $D = \{d\}$. We have $C \cap D = \emptyset$, so

$$f(C \cap D) = \emptyset \neq \{y\} = f(C) = f(D) = f(C) \cap f(D).$$

This directly contradicts (4), and we are done. ■