Math 424 B / 574 B Due Wednesday, Oct 07
Autumn 2015

Selected solutions for Homework 1

Solution for Problem 5, page 22. I need to show that if A is a non-empty bounded-below set of real numbers, then

$$\inf A = -\sup(-A),$$

where $-A$ is the set of all $-x$ for $x \in A$.

Let $x = \inf A$ and $y = \sup(-A)$. Since y is an upper bound for $-A$, we have

$$y \geq -a \Rightarrow -y \leq a, \ \forall a \in A,$$

so $-y$ is a lower bound for A. But x is the *greatest* lower bound for A, so we get

$$-y \leq x. \tag{1}$$

On the other hand, x is a lower bound for A, so

$$x \leq a \Rightarrow -x \geq -a \forall a \in A,$$

which means that $-x$ is an upper bound for $-A$. But y is the *least* upper bound for $-A$, and we have

$$y \leq -x \Rightarrow -y \geq x. \tag{2}$$

(1) and (2) show that the two numbers x and $-y$ are equal, which is what we wanted. ■

Solution for Problem 9, page 22. We have a binary relation $<$ on the set $\mathbb{C}$ of complex numbers, and we have to check that it's an order. This means that it has to satisfy conditions (i) and (ii) in Definition 1.5.

Checking condition (i). Let

$$x = a + bi, \ y = c + di$$

be two complex numbers. There are three possibilities for a and c:

One case is $a < c$ for the usual order on the set of real numbers, in which case $x < y$ for the order defined on $\mathbb{C}$ in the text of the problem.

Another case is $a > c$, which means $x > y$ as well, again by the definition of $>$ on $\mathbb{C}$.

Finally, you could have $a = c$. This splits into three further cases:

$b < d$, which means $x < y$;

$b > d$, which means $x > y$;

$b = d$; since we are also in the case $a = c$, this means $x = y$.

All of these cases are mutually exclusive, so we've checked that exactly one of the possibilities

$$x < y, \ x = y, \ x > y$$

holds.

Checking condition (ii). Suppose $x < y$ and $y < z$ for some

$$x = a + bi,\ y = c + di,\ z = e + fi \in \mathbb{C}.$$

Then, either (1) $a < c$ or (2) $a = c$ and $b < d$.

Similarly, there are two possibilities for y and z: either (1') $c < e$ or (2') $c = e$ and $d < f$.

Now check that $x < z$ in all four possible cases: (1)+(1'), (1)+(2'), (2)+(1') and (2)+(2').

Finally, the problem asks whether the order $<$ on $\mathbb{C}$ has the least upper bound property. I claim it does not. To prove this, I need to show you a subset of $\mathbb{C}$ bounded above which does not have an upper bound.

Consider the following subset:

$$A = \{bi : \ b \in \mathbb{R}\}.$$

So in other words, this is the set of those complex numbers $a + bi$ whose first component a is zero.

First, I claim it's bounded above. Indeed, the number $1 = 1 + 0i$ is an upper bound for it. Next, I claim it has no *least* upper bound. To see this, notice first that the sett of upper bounds for A is

$$B = \{a + bi : a > 0\}.$$

There is no smallest element in B, because for any element $a + bi \in B$ the element $\frac{a}{2} + bi$ is still in B but is strictly smaller than $a + bi$. ■

Extra Problem 1. Find the supremum and infimum of the sets

$$A = \left\{ \frac{(-1)^n}{n^2} : \ n = 1, 2, 3, \dots \right\}, \quad B = \{x \in \mathbb{R} : \ 2x^2 - 5x + 1 < 0\}$$

and

$$C = \{3^{-n} + 7^{-m} : \ m, n = 1, 2, 3, \dots\}.$$

Solution. We'll do these in turn.

Set A**.** For odd n the elements $\frac{(-1)^n}{n^2}$ are negative, while for even n they are positive. Moreover the absolute values

$$\left| \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$$

keep decreasing as n increases, so you'll get the largest possible element of A for the smallest even n, which is $n = 2$, and the smallest possible element of A for the smallest odd n, which is $n = 1$. In other words, we have

$$\inf A = \frac{(-1)^1}{1^2} = -1, \quad \sup A = \frac{(-1)^2}{2^2} = \frac{1}{4}.$$

Set B**.** First solve the equation $2x^2 - 5x + 1 = 0$ to see what the zeros of that polynomial are. You'll find the two solutions

$$x_\pm = \frac{5 \pm \sqrt{17}}{4}.$$

The graph of $f(x) = 2x^2 - 5x + 1$ is a parabola intercepting the x-axis are the two points x_- and x_+, and increasing to infinity both at $x \to -\infty$ and $x \to \infty$. So the set B where the value of $f(x)$ is negative is exactly the interval (x_-, x_+), which means that

$$\inf B = x_- = \frac{5-\sqrt{17}}{4}, \quad \sup B = x_+ = \frac{5+\sqrt{17}}{4}$$

Set C. When either m or n increases the value $3^{-n} + 7^{-m}$ decreases, so you'll get the largest element of C at $m = n = 1$; that's $\frac{1}{3} + \frac{1}{7} = \frac{10}{21}$.

On the other hand, as $m, n \to \infty$ the values get ever smaller and in fact approach 0 arbitrarily closely. That means that any lower bound for C must be ≤ 0. On the other hand, all elements of C are positive, so 0 is a lower bound. Al in all, we have

$$\inf C = 0, \quad \sup C = \frac{10}{21}.$$

■

Extra Problem 2. Let A and B be two non-empty sets of real numbers, and denote by $A + B$ the set $\{a + b : \ a \in A, \ b \in B\}$.

Show that $\sup(A + B) = \sup A + \sup B$.

Solution. Just as in the solution to Problem 5 above, we'll prove that each of the two numbers is at least as large as the other one, so that they must be equal.

Let $\alpha = \sup A$, $\beta = \sup B$, and $s = \sup(A + B)$.

Proof that $s \leq \alpha + \beta$. This was the easier of the two inequalities. Let $x \in A$ and $y \in B$ be two arbitrary elements. Then $\alpha \geq x$ and $\beta \geq y$ because α and β are upper bounds for A and B respectively, so $\alpha + \beta \geq x + y$. Since this happens for all possible choices of $x \in A$ and $y \in B$, the number $\alpha + \beta$ must be an upper bound for $A + B$. But s was by definition the *least* upper bound for the same set, so $s \leq \alpha + \beta$, as claimed.

Proof that $s \geq \alpha + \beta$. We will do this by contradiction. Suppose it's *not* the case that $s \geq \alpha + \beta$. Then, we must have $s < \alpha + \beta$. Let $\varepsilon > 0$ be a positive number so small that

$$2\varepsilon < \alpha + \beta - s. \tag{3}$$

It's possible to choose such a number, since my assumption is that $\alpha + \beta - s$ is strictly positive.

Putting the s on the left and the 2ε on the right in (3), I get

$$s < (\alpha - \varepsilon) + (\beta - \varepsilon).$$

Remember that α and β were the *least* upper bounds of A and B respectively. This means that $\alpha - \varepsilon$ and $\beta - \varepsilon$ are *not* upper bounds for A and B, so I can find

$$A \ni a > \alpha - \varepsilon \text{ and } B \ni b > \beta - \varepsilon. \tag{4}$$

Adding up these inequalities yields

$$A + B \ni a + b > (\alpha - \varepsilon) + (\beta - \varepsilon),$$

which in turn is greater than s by (4). But this now contradicts the fact that s was an upper bound for $A + B$.

We have our contradiction, which means that our assumption $s < \alpha + \beta$ must be wrong. This gets us $s \geq \alpha + \beta$, which is what was needed. ■