Math 424 B / 574 B
Autumn 2015

Two characterizations of compactness

The purpose of this note is to prove two results that I mentioned in class, on characterizing compactness via other properties we learned about.

1. Convergent subsequences

The first result, which I stated without proof, is

Theorem 1. *A metric space (X, d) is compact if and only if every sequence in X has a convergent subsequence.*

Before going into the proof, recall the following theorem from the previous note posted on the class website.

Theorem 2. *A metric space X is compact if and only if every infinite subset of X has a limit point.* ■

Proof of Theorem 1. We'll use Theorem 2 and substitute that property (every infinite subset has a limit point) for compactness. The goal is then to show that this property is equivalent to every sequence having a convergent subsequence.

Suppose every infinite subset of X has a limit point. Now let $(x_n)_n$ be a sequence in X. There are two possibilities: either the set $\{x_n\}$ is infinite, or it isn't.

Case 1: The set $\{x_n\}$ is finite. Then, infinitely many terms of the sequence, say x_{n_1}, x_{n_2}, etc. for $n_1 < n_2 < \cdots$, will be equal. But then the subsequence consisting of these terms is constant and hence convergent.

Case 2: The set $\{x_n\}$ is infinite. By assumption, the set will then have a limit point x. We proved in class that the set of limit points of the range $\{x_n\}$ of the sequence is contained in the set of limits of subsequences of $(x_n)_n$. This means that there exists a subsequence of (x_n) that converges to x, and we are done with case 2.

Cases 1 and 2 ensure that I can extract a convergent subsequence from any sequence, so we are finished with one implication.

Conversely, assume every sequence in X has a convergent subsequence. I now have to show that every infinite subset $S \subseteq X$ has a limit point. Since S is infinite, I can select distinct elements $x_n \in S$ for positive integers n.

The points x_n make up a sequence, which has a convergent subsequence $y_k = x_{n_k}$ by assumption. Let y be the limit of $(y_k)_k$. For any δ we can find some N such that

$$d(x_{n_k}, y) = d(y_k, y) < \delta$$

for all $k \geq N$. Since the points y_k are all distinct (because they are among the distinct points x_n), there are infinitely many y_ks in the neighborhood $N_\delta(y)$. In particular, there is at least one that is different from y. Since $\delta > 0$ was arbitrary, this means that y is a limit point of $\{y_k\}$ and hence also of the larger set S.

We have shown that every infinite subset of X has at least one limit point, so we are done with the second implication and the theorem. ∎

2. Completeness

Another characterization of compactness that I stated in class had to do with both completeness and total boundedness. Let us recall the relevant definitions.

Definition 3. A metric space (X, d) is *totally bounded* if for every $\delta > 0$ the space X can be covered by finitely many neighborhoods of radius δ. ♦

Definition 4. A metric space is *complete* if every Cauchy sequence is convergent. ♦

With this in place, we can now state the result.

Theorem 5. *A metric space is compact if and only if it is complete and totally bounded.*

Proof. Let (X, d) be a metric space.

(⇒) Here we are assuming X is compact and showing that it must then be both complete and totally bounded. You proved on your midterm that compact implies totally bounded (or at any rate the solutions posted on this website prove that). To show that compactness also implies completeness we proceed as follows.

Let $(x_n)_n$ be a Cauchy sequence in X. By compactness and Theorem 1 we know that (x_n) has a subsequence $y_k = x_{n_k}$ converging to some $y \in X$. By Problem 20 on page 82 of our book, the fact that (x_n) is Cauchy then implies the whole sequence converges to y. In particular the Cauchy sequence (x_n) is convergent, and we are done with the proof of the implication ⇒.

(⇐) This time around we are assuming that X is both complete and totally bounded, and seeking to prove that it is compact.

According to Theorem 1, it is enough to show that an arbitrary sequence in X has a convergent subsequence. In fact, since by completeness Cauchy sequences are convergent, it actually suffices to show that every sequence has a *Cauchy* subsequence. This is what Lemma 6 below does. ∎

Lemma 6. *Any sequence (x_n) in a totally bounded metric space (X, d) has a Cauchy subsequence.*

Proof. We will construct a Cauchy subsequence $y_k = x_{n_k}$ for $n_1 < n_2 < \cdots$ recursively as follows.

First, cover X with finitely many neighborhoods of radius $\frac{1}{2}$ (possible, by total boundedness). Because there are only finitely many of them, one of these neighborhoods will contain infinitely many terms x_n of the sequence. Let $I_1 \subseteq \mathbb{Z}_{>0}$ be an infinite set of indices such that all x_n, $n \in I_1$ are contained in the same radius-$\frac{1}{2}$ neighborhood, and choose $n_1 \in S_1$ arbitrarily.

For our second step in the recursion, cover X with finitely many neighborhoods of radius $\frac{1}{4} = \frac{1}{2\cdot 2}$. One of these neighborhoods contains infinitely many x_n for $n \in S_1$, and we denote by $S_2 \subseteq S_1$ an infinite set of indices n for which x_n belongs to this neighborhood. Then we select some $n_2 > n_1$ in S_2.

We continue the recursion as expected: in step $k \geq 3$ we cover X with finitely many neighborhoods of radius $\frac{1}{2\cdot k}$, denote by $S_k \subset S_{k-1}$ an infinite set of indices n such that x_n is contained in this neighborhood, and select some $n_k \in S_k$ bigger than all previously-selected indices (i.e. $n_{k-1} > \cdots > n_1$).

For all positive integers M and all $k, \ell \geq M$ the indices n_k and n_ℓ are by construction in S_M. This means that x_{n_k} and x_{n_ℓ} are in the same radius-$\frac{1}{2M}$ neighborhood $N_{\frac{1}{2M}}(x)$ for some $x \in X$, and hence

$$d(y_k, y_\ell) = d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, x) + d(x_{n_\ell}, x) < \frac{1}{2M} + \frac{1}{2M} = \frac{1}{M}.$$

In other words, for any M the terms y_k and y_ℓ are $\frac{1}{M}$-close to one another whenever k and ℓ are large enough. This is what it means for a sequence to be Cauchy, which concludes the proof of the lemma. ■