Math 424 B / 574 B
Autumn 2015

Some notes on compactness

This short write-up is meant to supplement the discussion on compactness in Chapter 2 of our textbook. Specifically, the main result below (Theorem 3) is kind of parallel to Theorem 2.41. I will expand on this after the proof of Theorem 3.

Before stating the theorem, a word of caution. In both the textbook and in class we have talked about compactness as a relative notion. That is, we talk about subsets of a metric space being compact. In fact, this complicates the discussion unnecessarily. As the paragraph following Theorem 2.33 notes, while being open or closed only makes sense relatively (i.e. you are open or closed as a subset of some ambient metric space), being compact makes sense on its own.

More precisely, say (X, d) is a metric space and $A \subseteq X$ is a subset. Then, I can restrict the metric d to A and think of (A, d) as a metric space in its own right. On the one hand, I can ask

Question 1. *Is $A \subseteq X$ is compact?*

This meant that out of every open subcover $A \subseteq \bigcup_\alpha G_\alpha$ of A in X I can extract a finite subcover. On the other hand, I can ask

Question 2. *Is (A, d) itself compact on its own?*

According to the definition, this would mean that out of every cover $A = \bigcup_\beta U_\beta$ with sets $U_\beta \subseteq A$ that are open with respect to the metric space structure (A, d) I can extract a finite subcover.

Now, it's is important to understand that the open subsets $U \subseteq A$ with respect to the metric space structure (A, d) *are exactly the intersections* $A \cap G$ for open subsets $G \subseteq X$ (this is essentially what Theorem 2.30 says).

In conclusion, Questions 1 and 2 are equivalent: the answer to both will be the same, so that from now on I will freely talk about compact metric spaces in isolation whenever convenient ('in isolation' as in, not necessarily viewed as subsets of some larger ambient metric space).

OK, now on to the theorem.

Theorem 3. *For any metric space (X, d) the following conditions are equivalent:*

(a) *X is compact.*
(b) *every infinite subset of X has a limit point.*

Proof. We will prove the two implications separately.

(a) $\Rightarrow$ (b). We pretty much did this in class on Monday, Oct 19, but I will type it again here in very slightly rephrased form. We prove the contrapositive. That is, if I can find an infinite subset $S \subseteq X$ that does *not* have a limit point, then X cannot be compact.

Let $S \subseteq X$ be as in the previous paragraph. Then by the definition of a limit point, for every $x \in X$ there is a neighborhood G_x of x that contains no elements of S except perhaps for x itself.

Now, every $x \in X$ is covered by G_x, so the family $\{G_x\}_{x\in X}$ is an open cover of X. I claim I cannot extract a finite subcover from it. This is because by our choice of G_x, each contains at most one element of S. This means that any finite collection of G_xs will cover only finitely many points in S, and so cannot cover all of $S \subseteq X$.

In conclusion, the assumption that (X, d) violates (b) leads to the conclusion that it doesn't satisfy (a) either, meaning that indeed (a) implies (b).

(b) $\Rightarrow$ (a). This was the more difficult implication.

We assume (X, d) satisfies (b), and fix an arbitrary open cover $\{G_\alpha\}_\alpha$ of X. We have to show that a finite subcover can be extracted. Before getting into that though, I will reindex the family $\{G_\alpha\}$ of open sets to make it more convenient, as follows.

By the definition of a cover, every $x \in X$ is contained in some G_α. Pick one such G_α for each x, and call it G_x. So now my family of open sets is $\{G_x\}_{x\in X}$, and for all x we have $x \in G_x$.

Out of the open cover $\{G_x\}$ I want to extract a finite subcover. We will first take a partial step in that direction:

Step 1: $\{G_x\}$ admits a subcover that is at most countable. For every positive integer n, I will say that a point $x \in X$ is *n-distant* (relative to the cover $\{G_x\}$) if the neighborhood $N_{\frac{1}{n}}(x)$ is contained in G_x (so think of it as saying that x is at least $\frac{1}{n}$ away from the "boundary" of G_x).

Now let X_n be the set of all n-distant points. Note that for every x *some* neighborhood $N_{\frac{1}{n}}(x)$ is contained in G_x (for sufficiently large n), so $X = \bigcup_n X_n$.

Claim: X_n can be covered by only finitely many sets G_x for n-distant x. Suppose not. Let x_1 be some n-distant point. We are assuming that G_{x_1} does not cover all of X_n, so there must be some n-distant $x_2 \in X_n \setminus G_{x_1}$. Again, we are assuming that

$$X_n \not\subseteq G_{x_1} \cup G_{x_2},$$

so there is some n-distant point

$$x_3 \in X_n \setminus (G_{x_1} \cup G_{x_2}).$$

Continue this recursive process, finding, for each positive integer $k > 1$, an n-distant point

$$x_k \in X_n \setminus \left(G_{x_1} \cup \cdots \cup G_{x_{k-1}}\right).$$

The set $\{x_k\}$ is infinite, because $x_k \in G_{x_k}$ but x_ℓ are by choice outside of G_{x_k} for $\ell > k$.

By condition (b) in the statement of the theorem, the infinite set $\{x_k\}$ has a limit point, called, say, x. But then, by Theorem 2.20, the neighborhood $N = N_{\frac{1}{2n}}(x)$ contains infinitely many of the x_k. This is not possible for the following reason.

Since $x_k \in G_k$ is n-distant, we have

$$N_{\frac{1}{n}}(x_k) \subseteq G_{x_k}.$$

On the other hand, for all $\ell > k$ the point x_ℓ is outside of G_{x_k} by choice and hence outside of $N_{\frac{1}{n}}(x_k)$ as well. In other words, we have

$$d(x_k, x_\ell) \geq \frac{1}{n},\ \forall k \neq \ell. \tag{1}$$

Now, if two distinct x_k and x_ℓ were to belong to N, we would have

$$d(x_k, x_\ell) \le d(x, x_k) + d(x, x_\ell) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

contradicting (1).

We have reached a contradiction, which can only stem from our assumption that X_n could *not* be covered by finitely many G_x for n-distant x. This finishes the proof of the claim.

Now extract an at-most-countable subcover from $\{G_x\}$ as follows. For each n, keep only finitely many sets G_x for n-distant x, but enough to ensure that they cover X_n. This is possible by the claim.

You have kept finitely many G_xs for each n, so in total you have kept at most countably many G_xs. The sets that are left do indeed cover X, because you have made sure that for each n the set X_n is covered and we observed above that $X = \bigcup_n X_n$.

This concludes the proof of Step 1. To simplify notation, let $\{G_n\}$ be an open subcover extracted from $\{G_x\}$.

Step 2: The at-most-countable cover $\{G_n\}$ admits a finite subcover. Suppose not. Then, I can find elements $x_n \in X$ for $n = 1, 2, \cdots$ such that

$$x_n \notin G_1 \cup \cdots \cup G_n, \ \forall n > 0.$$

The x_n make up an infinite set (because x_n must be contained in *some* G_m, but the x_p, $p > m$ are not contained in that same G_m by assumption). Condition (b) of the statement of the theorem then says that the set $\{x_n\}$ should have a limit point, say x.

Since $x \in X$ and $\{G_n\}$ is a cover, we have $x \in G_m$ for some m, and hence there is some neighborhood $N = N_r(x) \subseteq G_m$. But then, for $n > m$, $x_n \notin N$, contradicting the fact that every neighborhood of x should contain infinitely many x_n (Theorem 2.20).

We have reached a contradiction based on the assumption that there was *not* a finite subcover of $\{G_n\}$, so such a finite subcover must exist. This concludes the proof of Step 2 and the theorem. ■

Now, how does Theorem 3 relate to Theorem 2.41? I mentioned before that they are incomparable, in the sense that neither is stronger than the other. Theorem 2.41 says more in the sense that in addition to our conditions (a) and (b) (which in 2.41 correspond to (b) and (c) respectively) it also throws the condition of being both bounded and closed in $\mathbb{R}^k$ into the mix. The hypotheses of 2.41 however are correspondingly stronger, making it more restrictive: it only talks about subsets of $\mathbb{R}^k$, whereas Theorem 3 is about arbitrary metric spaces.

So: Theorem, 2.41 assumes more but also says more than Theorem 3.

Note also that although conditions (b) and (c) are absolute in nature, in the sense that you can talk about them being satisfied by E as a metric space in its own right, with the metric inherited from $\mathbb{R}^k$ (so the book's E is like our X in Theorem 3), condition (a) is relative, i.e. it needs an ambient metric space because it makes use of the notion of being closed.

As noted in the discussion after 2.41, the metric spaces $\mathbb{R}^k$ are very special in one respect: closed bounded subsets are compact. Because this result is, as mentioned in the textbook, due to two mathematicians named Heine and Borel, the following definition has become standard.

Definition 4. A metric space has *the Heine-Borel (or H-B) property* or is *Heine-Borel (or H-B)* if its closed bounded subsets are compact. ♦

The Heine-Borel property is why Theorem 2.41 needs to assume we are in $\mathbb{R}^k$: not all metric spaces are H-B. Some counterexamples are mentioned in the book, one of which is worked out as Exercise 16. Here are some more examples of metric spaces with and without the H-B property.

Example 5. Compact metric spaces are H-B. Indeed, every closed subset of a metric space is compact by Theorem 2.35. ♦

Example 6. The spaces $\mathbb{R}^k$ are H-B not only when equipped with the usual distance function coming from the norm $|\cdot|$ introduced in the book (in the section on Euclidean spaces in Chapter 1), but also with the *taxicab distance* d_1 defined by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} |x_i - y_i|$$

for

$$\mathbf{x} = (x_1, \cdots x_k), \ \mathbf{y} = (y_1, \cdots y_k)$$

and the *supremum distance* d_∞ defined by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{i=1}^{k} |x_i - y_i|$$

(try to prove these are distance functions). ♦

Example 7. Let X be any infinite set whatsoever, and equip it with the *discrete metric* d defined by $d(x, y) = 1$ whenever $x \neq y$ (and necessarily $d(x, x) = 0$, because I want it to be a metric).

Try to prove as an exercise that *all* subsets of X are open, and hence all subsets are also closed. On the other hand X is clearly bounded, since for any point x we have $X = N_2(x)$ (that is, all points $y \in X$ are less than 2 away from x). So every subset of X is closed and bounded, but only the finite subsets are compact (try to prove this last statement; it should be easy to do directly from the definition). ♦

Example 8. We can mimic the supremum distance d_∞ from Example 6 in an "infinite-dimensional" setting as follows.

Let ℓ^∞ denote the set of bounded sequences

$$\mathbf{x} = (x_1, x_2, x_3, \cdots)$$

of real numbers, and define

$$d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{i \geq 1} |x_i - y_i|.$$

Once more this can be shown to be a distance function, so that (ℓ^∞, d_∞) is a metric space. I claim it does not have the H-B property.

To see this, consider the set

$$E = \{e_n : \ n = 1, 2, \cdots\},$$

where e_n is the sequence (x_k) with $x_n = 1$ and $x_m = 0$ for $m \neq n$. So

$$e_1 = (1, 0, 0, \cdots),$$

$$e_2 = (0, 1, 0, \cdots),$$

and so on

First, the set is bounded because the distance between any two e_ns is 1. On the other hand, E is also closed. In fact, we have

Lemma 9. *The subset $E \subset \ell^\infty$ has no limit points.*

Proof. If it did have a limit point x say, the neighborhood $N = N_{\frac{1}{2}}(x)$ would contain infinitely many points of E by 2.20. But this is impossible: as noted before, the distance between any two different e_ns is 1. If say e_m and e_n were both inside N, we would have

$$d(e_m, e_n) \leq d(x, e_n) + d(x, e_m) < \frac{1}{2} + \frac{1}{2} = 1,$$

contradicting $d(e_m, e_n) = 1$. ■

Having no limit points is definitely sufficient to ensure that E is closed (because E' is empty, so $\overline{E} = E \cup E' = E$). But it also ensures that E is not compact: it's an infinite set with no limit point, violating condition (b) in Theorem 3. ♦