Math 424 B / 574 B Due Friday, Dec 11
Autumn 2015

Solution to Bonus Homework 2

As with the previous extra homework, there is no penalty if you do not turn it in, and no partial credit. If answered correctly with correct justification the question is worth 1% extra credit.

1. Does there exist a function $f : \mathbb{Q} \cap (0, \infty) \to \mathbb{R}$ such that $\lim_{x\to p} f(x) = \infty$ for all positive rational numbers p?

Solution. The answer was 'yes': such a function exists. The hint I sent you by email, saying that you should hijack one of the problems at the end of Chapter 4, referred to Problem 18 (page 100). Here's what I meant by that.

Every positive rational $x \in \mathbb{Q} \cap (0, \infty)$ can be written uniquely as $\frac{m}{n}$, where m and n are positive integers with no common divisors > 1 (just as in Problem 18). I will call two such positive integers *coprime*. So for instance 2 and 9 are coprime because no positive integer divides them both except for 1, but 6 and 9 are not coprime because 3 divides both.

Now simply set

$$f\left(\frac{m}{n}\right) = n$$

My claim now is that indeed

$$\lim_{x\to p} f(x) = \infty$$

for all $p \in \mathbb{Q} \cap (0, \infty)$. To see this, argue as follows.

Fix your positive rational $p = \frac{u}{v}$ for coprime positive integers u, v. For any positive integer $n > v$, the rationals of the form $\frac{m}{n}$ for coprime m, n with no common divisors cannot approach p arbitrarily well, because

$$0 < \left|p - \frac{m}{n}\right| = \left|\frac{u}{v} - \frac{m}{n}\right| = \frac{|un - vm|}{vn},$$

which is at least $\frac{1}{vn}$ (because it's a strictly positive rational with denominator vn). So if you approach p arbitrarily closely with positive rationals, the denominators n in the coprime fraction expression $\frac{m}{n}$ of these rationals must increase to infinity. This ends the proof. ■