Math 424 B / 574 B Due Friday, Nov 06
Autumn 2015

Solutions for bonus homework

These questions are meant to get you to explore the notion of perfect set a bit, getting you some additional points towards your course grade along the way.

There is no penalty if you do not turn them in, and no partial credit: each of the three questions, if answered correctly with correct justification, is worth 0.5% extra credit.

1. Does there exist a non-empty perfect subset of $\mathbb{R}$ that contains no rational numbers?

2. Same as above, with 'irrational' instead of 'rational'.

3. Same as before, with 'algebraic' instead of '(ir)rational'.

Part 2. This would have been the easiest one to answer: 'no'.

A set as in the text of the problem would have to be contained in $\mathbb{Q}$, and hence be at most countable. But according to Theorem 2.43, non-empty perfect subsets of $\mathbb{R}$ are uncountable, so this cannot happen. ■

Parts 1 and 3. These had a common answer: 'yes'. Both $\mathbb{Q}$ and the set of algebraic numbers are countable (the second one by Problem 2 on page 43). So the positive answer follows from

Proposition 1. *For any countable subset $S \subset \mathbb{R}$, there is a non-empty perfect subset of $\mathbb{R}$ contained in $S^c = \mathbb{R} \setminus S$.*

Proof. Enumerate the set S as

$$S = \{s_1, s_2, \cdots\}.$$

Now define the sets $T_0 \supset T_1 \supset \cdots$ recursively as follows.

First, take $T_0 = \mathbb{R}$.

Next, let $I_1 = (a_1, b_1)$ be an open interval containing s_1 and such that $a_1, b_1 \notin S$, and set $T_1 = T_0 \setminus I_1$.

Next, let $I_2 = (a_2, b_2) \subset T_1$ be an open interval containing s_2, with $a_2, b_2 \notin S$, and such that either $I_1 = I_2$ (if s_2 was already in I_1), or the four endpoints of I_1 and I_2 are all distinct if $s_2 \notin I_1$. Then set $T_2 = T_1 \setminus I_2$.

Now continue this procedure recursively:

At step $n \geq 3$ choose an interval $I_n = (a_n, b_n) \subset T_{n-1}$ that contains s_n, such that either I_n coincides with one of the previous intervals I_1 up to I_{n-1} if their union already contains s_n, or the endpoints of I_n are distinct from those of all previous I_j, $1 \leq j \leq n-1$. Then set $T_n = T_{n-1} \setminus I_n$

By construction the intersection $T = \bigcap_{n>0} T_n$ does not intersect S (because at each step we are removing one more point $s_n \in S$), and you can show just as for the Cantor set that T is perfect. ■

Proposition 1 applies in particular to $S = \mathbb{Q}$ and S the set of algebraic numbers, so we are done. ■