Solutions for bonus homework

These questions are meant to get you to explore the notion of perfect set a bit, getting you some additional points towards your course grade along the way.

There is no penalty if you do not turn them in, and no partial credit: each of the three questions, if answered correctly with correct justification, is worth 0.5% extra credit.

1. Does there exist a non-empty perfect subset of $\mathbb{R}$ that contains no rational numbers?

2. Same as above, with ‘irrational’ instead of ‘rational’.

3. Same as before, with ‘algebraic’ instead of ‘(ir)rational’.

Part 2. This would have been the easiest one to answer: ‘no’.

A set as in the text of the problem would have to be contained in $\mathbb{Q}$, and hence be at most countable. But according to Theorem 2.43, non-empty perfect subsets of $\mathbb{R}$ are uncountable, so this cannot happen.

Proof. Enumerate the set $S$ as $S = \{s_1, s_2, \cdots \}$.

Now define the sets $T_0 \supset T_1 \supset \cdots$ recursively as follows.

First, take $T_0 = \mathbb{R}$.

Next, let $I_1 = (a_1, b_1)$ be an open interval containing $s_1$ and such that $a_1, b_1 \notin S$, and set $T_1 = T_0 \setminus I_1$.

Next, let $I_2 = (a_2, b_2) \subset T_1$ be an open interval containing $s_2$, with $a_2, b_2 \notin S$, and such that either $I_1 = I_2$ (if $s_2$ was already in $I_1$), or the four endpoints of $I_1$ and $I_2$ are all distinct if $s_2 \notin I_1$. Then set $T_2 = T_1 \setminus I_2$.

Now continue this procedure recursively:
At step $n \geq 3$ choose an interval $I_n = (a_n, b_n) \subset T_{n-1}$ that contains $s_n$, such that either $I_n$ coincides with one of the previous intervals $I_1$ up to $I_{n-1}$ if their union already contains $s_n$, or the endpoints of $I_n$ are distinct from those of all previous $I_j$, $1 \leq j \leq n - 1$. Then set $T_n = T_{n-1} \setminus I_n$.

By construction the intersection $T = \bigcap_{n \geq 0} T_n$ does not intersect $S$ (because at each step we are removing one more point $s_n \in S$), and you can show just as for the Cantor set that $T$ is perfect. ■

Proposition 1 applies in particular to $S = \mathbb{Q}$ and $S$ the set of algebraic numbers, so we are done. ■