Math 424 B / 574 B
Autumn 2015

A proof that $\mathbb{R}$ is uncountable

This is a write-up of the proof we did in class for the following result.

Theorem 1. *The set $\mathbb{R}$ of all real numbers is uncountable.*

This boiled down to showing that $\mathbb{R}$ is not countable, i.e. there is no bijection $f : \mathbb{Z}_{>0} \to \mathbb{R}$. Before going into the proof, I'll need some preparation.

First, remember that for real numbers $a < b$ we're denoting by (a, b) the *open interval*

$$\{x \in \mathbb{R} : \ a < x < b\}.$$

I'll need the following notion.

Definition 2. Two open intervals

$$(a, b) = I \subset J = (c, d)$$

are *strictly nested* if $a > c$ and $b < d$. ♦

Remark 3. In other words, one is contained in the other and all four of their endpoints are distinct. ♦

We first proved an auxiliary result.

Lemma 4. *If $I_1 \supset I_2 \supset I_3 \supset \cdots$ is a sequence of strictly nested open intervals, then the intersection*

$$\bigcap_{n \geq 1} I_n \subset \mathbb{R} \tag{1}$$

is non-empty.

Proof. Let $I_n = (a_n, b_n)$. I'll define the sets

$$A = \{a_n : \ n > 0\} \quad \text{and} \quad B = \{b_n : \ n > 0\}.$$

Note that every element of B is greater than every element of A, so that A is bounded above. It's also non-empty, so it has a supremum. Denote $a = \sup A$.

Claim: $a \in I_n$ **for every** $n > 0$**.** Since a is an upper bound for A, we have

$$a_n \leq a, \ \forall n > 0.$$

We cannot have equality in any of these inequalities, because if say $a_n = a$ then $a_{n+1} > a_n = a$ (this is where the hypothesis of being strictly nested is used). In conclusion,

$$a_n < a, \ \forall n > 0. \tag{2}$$

On the other hand, all b_n are upper bounds for A. Since a is the *least* upper bound, we get

$$a \leq b_n, \ \forall n > 0.$$

Once more, we cannot have equality in any of these: if $a = b_n$, then $b_{n+1} < b_n = a$ (once more the condition of being strictly nested is needed for the inequality $b_{n+1} < b_n$). So we have

$$a < b_n,\ \forall n > 0. \tag{3}$$

Equations (2) and (3) together say that $a \in (a_n, b_n) = I_n$ for all $n > 1$.

This proves the claim. Since we've exhibited an element a that belongs to the intersection (1), this concludes the proof of the lemma. ■

Finally, we can get down to business.

Proof of Theorem 1. We proceed by contradiction: suppose *there is* a bijection $f : \mathbb{Z}_{>0} \to \mathbb{R}$. For $k \in \mathbb{Z}_{>0}$ I'll denote $f(k)$ by r_k.

Our assumption means that r_k exhaust all of $\mathbb{R}$ as k ranges over the positive integers. I will construct a sequence $I_1 \supset I_2 \supset \cdots$ of strictly nested open intervals to which we will then apply Lemma 4. I have to say what the n^{th} interval $I_n = (a_n, b_n)$ is. I will select the endpoints a_n and b_n from among the r_ks recursively as follows.

- Set $a_1 = r_1$;
- Let b_1 be the earliest r_k (that is, the one with the smallest index n) such that $a_1 < r_k$;
- Let a_2 be the earliest r_k such that $a_1 < r_k < b_1$;
- Let b_2 be the earliest r_k such that $a_2 < r_k < b_1$;
- $\cdots$

In other words, at every step choose the earliest r_n that will satisfy the inequalities you need in order that the intervals $I_n = (a_n, b_n)$ be strictly nested.

Note that it's always possible to find the r_k you need, because at each step you're looking for the earliest r_k that is strictly between two numbers $a_i < b_j$ that you've already chosen. This is possible for example because there are always rational numbers strictly between any two distinct real numbers (this is part (b) of Theorem 1.20 in your book).

Once I have the sequence of strictly nested open intervals $\{I_n\}$, I know from Lemma 4 that there is a real number

$$r \in \bigcap_{n>0} I_n.$$

Claim: $r \neq r_k$ for all $k > 0$. To see this, let us examine the construction of the a_ns and b_ns.

Remember that we were selecting the as and bs by choosing from among the r_ks. I'll denote the m^{th} r_k selected in that recursive process by r_{k_m}. So $r_{k_1} = a_1$, $r_{k_2} = b_1$, $r_{k_3} = a_2$, etc. In other words, r_{k_m} corresponds to the m^{th} bullet point in the above description of the recursion.

First, $r_{k_1} = r_1 = a_1$ cannot be r, because $r \in I_1$ and so is greater than the left hand endpoint a_1 of $I_1 = (a_1, b_1)$.

Next, remember that b_1 was the *earliest* r_k that is bigger than a_1. This means that none of the rs that are listed between r_{k_1} and $r_{k_2} = b_1$ are bigger than a_1, so none of them are in I_1. Consequently, none of them can coincide with $r \in I_1$.

Next, a_2 was the earliest r_k caught between a_1 and b_1, so none of the r_ks listed between $r_{k_2} = b_1$ and $r_{k_3} = a_2$ are in the interval (a_1, b_1), so none of them coincide with r.

Continue this argument inductively to conclude that for all m, the r_ks listed between r_{k_m} and $r_{k_{m+1}}$ are all different from r. This proves the claim.

Since we've come up with a real number r that is different from all r_n, this contradicts our assumption that the r_ns were *all* of the real numbers. We have our contradiction, so we are done. ■