Midterm

Name: ____________________________________________

Student ID: _______________________________________

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- Please read this.
- You have 50 minutes.
- No notes or calculators.
- The problems are ordered randomly; I’d maybe read all of them first to find the easiest ones.
- Show all of your work.
- Use the reverse side of each page for extra space and add paper of your own if you run out.
Problem 1. (25 points)
Find a fundamental set of solutions for the system

\[ x' = \begin{pmatrix} 0 & 1 & 1 \\ -3 & -4 & -3 \\ 2 & 2 & 1 \end{pmatrix} x. \]

Hint: \( a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3 \)

Solution
The characteristic polynomial is the determinant

\[ \begin{vmatrix} \lambda & -1 & -1 \\ 3 & \lambda + 4 & 3 \\ -2 & -2 & \lambda - 1 \end{vmatrix}, \]

which works out to \( \lambda^3 + 3\lambda^2 + 3\lambda + 1 \). The hint was supposed to tip you off that this is \( (\lambda + 1)^3 \), which means \(-1\) is an eigenvalue of algebraic multiplicity 3.

To find the eigenvectors, I have to solve

\[ \begin{pmatrix} 1 & 1 & 1 \\ -3 & -3 & -3 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

The three equations are scalings of one another, so they’re all saying the same thing: \( v_1 + v_2 + v_3 = 0 \). This means that the eigenvectors are the (non-zero) vectors of the form

\[ \begin{pmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \]

so I get two solutions in the usual way:

\[ e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]

You’re in the case when the geometric multiplicity of the eigenvalue is two (you were able to find two linearly independent eigenvectors and no more), so the third solution should be of the form \( te^{-t}v + e^{-t}w \) for vectors \( v \) and \( w \) such that \( (A + 1)v = 0 \) and \( (A + 1)w = v \).

We reviewed in class the general procedure for how to come up with such a solution: Rather than trying to figure out which eigenvector \( v \) will make the
system \((A + 1)w = v\) consistent, it’s faster to take for \(w\) any vector that’s not of the form (1), and then simply set \(v\) to be \((A + 1)w\).

You are now free to choose \(w\) however you want, so long as its three components don’t add up to zero (so that it’s not as in equation (1)). For example, take \(w_1 = w_2 = 0\) and \(w_3 = 1\). Then

\[
(A + 1)w = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix},
\]

and

\[
t e^{-t} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

is a third solution of your system. Taking the three solutions you have as columns you get a fundamental matrix

\[
\Psi(t) = e^{-t} \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & -3t \\ -1 & -1 & 2t + 1 \end{pmatrix}.
\]

Your answer doesn’t have to be precisely this to get a full score (you could have chosen \(w\) differently for instance, or written down a different pair of eigenvectors than I did). So long as you have a correct fundamental matrix that you got through correct work I can check you’ll get a full score.
Problem 2. (20 points) Write down a real-valued fundamental matrix $\Psi(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix} \mathbf{x}.$$

Solution

The characteristic polynomial is $\lambda^2 + 2\lambda + 2$, so the roots are $1 \pm i$. You know from class that it’s enough for me to get one eigenvector for one of the eigenvalues, and that’ll tell me everything I want.

So let’s find an eigenvector $\mathbf{v}$ for $1 + i$. I’m solving

$$\begin{pmatrix} 2 - i & 1 \\ -5 & -2 - i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

so $v_2 = (-2 + i)v_1$. The eigenvectors, in other words, are scalar multiples of $\begin{pmatrix} 1 \\ -2 + i \end{pmatrix}$.

From all of this I get a solution to my system:

$$e^{(1+i)t} \begin{pmatrix} 1 \\ -2 + i \end{pmatrix} = e^t(\cos t + i \sin t) \begin{pmatrix} 1 \\ 2 - i \end{pmatrix} = e^t \begin{pmatrix} \cos t + i \sin t \\ -2 \cos t - \sin t + i(\cos t - 2 \sin t) \end{pmatrix}.$$

Now simply take the real and imaginary parts of that and treat them as columns to get your fundamental matrix:

$$\Psi(t) = e^t \begin{pmatrix} \cos t & \sin t \\ -2 \cos t - \sin t & \cos t - 2 \sin t \end{pmatrix}.$$
Problem 3. (20 points) Solve the initial value problem

\[ x' = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}. \]

Solution
I have to first find the general solution, and then use the initial value condition to single out a specific solution.

The characteristic polynomial of the coefficient matrix is \( \lambda^2 + 2\lambda - 3 \), so the eigenvalues are 1 and -3.

I’ll first find the eigenvectors for \( \lambda = 1 \), solving

\[ \begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}. \]

This means \( v_1 = v_2 \), so that eigenvectors are scalar multiples of \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

Similarly, for \( \lambda = -3 \) I have to solve

\[ \begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}. \]

This says that \( v_2 = 5v_1 \), so \( \begin{pmatrix} 1 \\ 5 \end{pmatrix} \) is an eigenvector.

In conclusion, the general solution is a linear combination

\[ x(t) = c_1e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \]

Imposing the additional condition \( x(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \) forces

\[
\begin{cases}
   c_1 + c_2 = -2 \\
   c_1 + 5c_2 = 2
\end{cases}
\]

Subtracting the first equation from the second one gives \( c_2 = 1 \), and substituting back into the first equation then gives \( c_1 = -3 \). The final answer is

\[ x(t) = -3e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \]
Problem 4. (25 points) Find a particular solution to
\[ x' = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} -2t \\ t \end{pmatrix}, \ t > 0. \]

Solution

The characteristic polynomial of the coefficient matrix is \( \lambda^2 \), so you have the eigenvalue 0 with algebraic multiplicity two.

To find eigenvectors, you want to solve
\[ \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \]

This means that \( v_1 = -2v_2 \), so \( \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) is the only eigenvector up to scaling. We are in the situation where the algebraic multiplicity is two but the geometric multiplicity is one, so I’ll continue by variation of parameters.

That method tells you that
\[ \Psi(t) \int \Psi(t)^{-1} b(t) \, dt \]
is a solution whenever \( \Psi(t) \) is a fundamental matrix of the homogeneous system with the same coefficient matrix (for \( b(t) = \begin{pmatrix} -2t \\ t \end{pmatrix} \)).

So let’s first find a fundamental matrix. From the eigenvector I can construct a first solution, \( \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) (this is just a constant vector function; the factor \( e^\lambda t \) is 1 because \( \lambda = 0 \)).

The general procedure for producing extra solutions says that you should now seek one of the form \( t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + w \), where \( w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \) is some vector such that \( A w = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \). This latter linear system reads
\[
\begin{cases}
2w_1 + 4w_2 = -2 \\
-w_1 - 2w_2 = 1
\end{cases}
\]

The two equations are just scalar multiples of one another, so they both mean the same thing: \( w_1 + 2w_2 = -1 \). Any \( w \) that satisfies this condition will do, so I might as well just take \( w = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \). I now have my second solution to the
homogeneous system:

$$t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + w = \begin{pmatrix} -2t - 1 \\ t \end{pmatrix}.$$ 

Putting the two solutions I’ve got together into a fundamental matrix yields

$$\Psi(t) = \begin{pmatrix} -2 & -2t - 1 \\ 1 & t \end{pmatrix}.$$ 

I need its inverse; using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

you can compute its inverse:

$$\Psi(t)^{-1} = \begin{pmatrix} t & 2t + 1 \\ -1 & -2 \end{pmatrix}.$$ 

The vector I have to integrate is

$$\Psi(t)^{-1} b(t) = \begin{pmatrix} t & 2t + 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2t \\ t \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$ 

You can write down an antiderivative for this immediately: $\begin{pmatrix} \frac{t^2}{2} \\ 0 \end{pmatrix}$. Finally, to get a solution to my original system I have to multiply this vector function on the left by the fundamental matrix. So a particular solution might be:

$$\Psi(t) \begin{pmatrix} \frac{t^2}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -t^2 \\ \frac{t^2}{2} \end{pmatrix} = t^2 \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}.$$