Surface area example

This note is meant to tie some loose ends left over from the lecture on Friday Oct 13 2017. Let \( f(x, y) \) be a function of two variables. First, recall that we decided that the area of the portion of the graph of \( f \) above the domain \( D \) in the plane is

\[
\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy. \tag{1}
\]

We then sought to verify that this does indeed give the correct answer by checking against the function

\[ f(x, y) = \sqrt{1 - x^2 - y^2} \]

over the domain

\[ D = \{(x, y) \mid x^2 + y^2 \leq 1\} \]

(this is simply the unit disk centered at the origin of the \( xy \) plane).

The partial derivatives of \( f \) are

\[ f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}, \]

so (1) then reads

\[
\iint_D \sqrt{1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}} \, dx \, dy.
\]

Writing \( 1 = \frac{1-x^2-y^2}{1-x^2-y^2} \), the integral becomes

\[
\iint_D \sqrt{1 - x^2 - y^2} \, dx \, dy,
\]

and we decided to compute it by passing to polar coordinates.

The bounds for \( r \) and \( \theta \) describing the unit disk are \([0, 1]\) and \([0, 2\pi]\) respectively, and \( x^2 + y^2 = r^2 \). Remembering that \( dx \, dy = r \, dr \, d\theta \), the same integral in polar coordinates is

\[
\int_0^{2\pi} \int_0^1 \frac{1}{1 - r^2} \, r \, dr \, d\theta.
\]

Since the integrand does not depend on \( \theta \), the outer integral simply multiplies the inner integral by \( 2\theta \), so that the answer is

\[ 2\pi \int_0^1 \frac{r}{1 - r^2} \, dr. \]
Finally, to compute this, note that the function \( \sqrt{\frac{r}{1-r^2}} \) being integrated is nothing but the derivative of \(-\sqrt{1-r^2}\), so the number we’re after is
\[
2\pi \left( -\sqrt{1-r^2} \right) \bigg|_0^1 = 2\pi.
\]

On the other hand, the graph of \( f \) is simply the upper hemisphere of the sphere of radius 1 centered at the origin. The area of a sphere of radius \( R \) is \( 4\pi R^2 \), so for \( R = 1 \) the area of a hemisphere is indeed \( 2\pi \).

In other words, the method of calculating areas we learned in class does check out in this example.