Finding potentials for vector fields

I'll illustrate here the process of finding a potential for a conservative vector field
\[ \langle P(x, y), Q(x, y) \rangle \]
in the plane. This is meant to fill in the piece of the Friday, Oct 20 lecture, missing because of the computational error I made.

The vector field I'll work with is
\[ \langle P(x, y), Q(x, y) \rangle = \langle \sin(x + y) + x \cos(x + y) + 2x, x \cos(x + y) \rangle. \]

Suppose we already know that it’s conservative. The problem is then to find a potential \( f(x, y) \) for it, i.e. a function satisfying the two equations
\[ f_x = P, \quad f_y = Q. \]

From the second equation \( f_y = x \cos(x + y) \), by integrating with respect to \( y \) (and keeping \( x \) constant) we obtain
\[ f(x, y) = x \sin(x + y) + C(x). \quad (1) \]

Note the +C(x) term! It’s like the +C from antiderivatives, except here we’re secretly doing many antiderivatives, one for each \( x \), and the corresponding constants \( C \) might depend on \( x \).

We now make use of the second piece of information we have about \( f \): the fact that \( f_x = P \).

Plugging (1) into \( f_x = P \) reads
\[ \sin(x + y) + x \cos(x + y) + C'(x) = \sin(x + y) + x \cos(x + y) + 2x, \]
so \( C'(x) = 2x \). This means that \( C(x) = x^2 + D \) for some constant \( D \). Plugging this back into (1) we finally get
\[ f(x, y) = x \sin(x + y) + x^2 + D \]
for some constant \( D \).

Note that potentials are never unique: if \( f \) is a potential then so is \( f(x + y) + D \) for any constant \( D \), so don’t forget that extra constant (just like the +C for antiderivatives).