ON THE MEMBERSHIP OF HANKEL OPERATORS IN A CLASS OF LORENTZ IDEALS

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Abstract. Recall that the Lorentz ideal $\mathcal{C}_p^-$ is the collection of operators A satisfying the condition $\|A\|_p^- = \sum_{j=1}^\infty j^{-(p-1)/p} s_j(A) < \infty$. Consider Hankel operators $H_f : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$, where $H^2(S)$ is the Hardy space on the unit sphere S in $\mathbf{C}^n$. In this paper we characterize the membership $H_f \in \mathcal{C}_p^-$, $2n < p < \infty$.

1. Introduction

The study of Hankel operators has a long and rich history [1,2,4-7,10-14,17-19]. We are particularly interested in one kind of Hankel operators: those on the Hardy space of the unit sphere. Let us begin by describing our basic setting.

Let S be the unit sphere $\{z : |z| = 1\}$ in $\mathbf{C}^n$. In this paper, the complex dimension n is always assumed to be greater than or equal to 2. Let $d\sigma$ be the standard spherical measure on S. That is, $d\sigma$ is the positive, regular Borel measure on S with $\sigma(S) = 1$ that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0.

Recall that the Hardy space $H^2(S)$ is the norm closure in $L^2(S, d\sigma)$ of the collection of polynomials in the complex variables $z_1, \dots, z_n$. As usual, we let P denote the orthogonal projection from $L^2(S, d\sigma)$ onto $H^2(S)$. The main object of study in this paper, the Hankel operator $H_f : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$, is defined by the formula

$$H_f = (1 - P)M_f | H^2(S).$$

To motivate what we will do in this paper, let us briefly review what has been done so far.

We consider symbol functions $f \in L^2(S, d\sigma)$. Recall that the problems of boundedness and compactness of H_f were settled in [17]. Later, in [5] we characterized the membership of H_f in the Schatten class $\mathcal{C}_p$, $2n < p < \infty$. Moreover, it was shown in [5] that the membership $H_f \in \mathcal{C}_{2n}$ implies $H_f = 0$. More recently, in [6] we characterized the membership of H_f in the ideal $\mathcal{C}_p^+$, $2n < p < \infty$.

In this paper, we turn our attention to the membership of H_f in the Lorentz ideal $\mathcal{C}_p^-$. Before going any further, it is necessary to recall the definition of these operator ideals.

Given an operator A, we write $s_1(A), \dots, s_j(A), \dots$ for its s-numbers [9,Section II.2]. For each $1 < p < \infty$, the formula

$$\|A\|_p^- = \sum_{j=1}^{\infty} \frac{s_j(A)}{j^{(p-1)/p}}$$

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defines a symmetric norm for operators [9,Section III.15]. On any separable Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^- = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^- < \infty\}$$

is a norm ideal [9,Section III.2].

Closely associated with the Lorentz ideals $\mathcal{C}_p^-$ are the ideals $\mathcal{C}_p^+$, which are defined as follows. For each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{j \geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

also defines a symmetric norm for operators [9,Section III.14]. On any separable Hilbert space $\mathcal{H}$, we have the norm ideal

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}.$$

As we mentioned, the $\mathcal{C}_p^+$'s were the ideals of interest in [6]. In this paper, these ideals will play an important supporting role.

Compared with the more familiar Schatten class $\mathcal{C}_p = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p < \infty\}$, where $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$, for all $1 < p' < p < \infty$ we have the relation

$$\mathcal{C}_{p'}^+ \subset \mathcal{C}_p^- \subset \mathcal{C}_p \subset \mathcal{C}_p^+,$$

with all the inclusions being proper. This explains the $+$ and $-$ in the notation: $\mathcal{C}_p^-$ is slightly smaller than $\mathcal{C}_p$, whereas $\mathcal{C}_p^+$ is slightly larger than $\mathcal{C}_p$.

Since the membership problem $H_f \in \mathcal{C}_p^+$, $2n < p < \infty$, was settled in [6], the obvious next step is to determine the membership $H_f \in \mathcal{C}_p^-$, $2n < p < \infty$. But this next step, however natural it is, turns out to be quite a challenge. We have a sizable collection of techniques from previous investigations [5,6,18], but these techniques alone are not sufficient for the membership problem $H_f \in \mathcal{C}_p^-$. The reason for that is that the norm $\|\cdot\|_p^-$ is much harder to work with than $\|\cdot\|_p^+$. But, with considerable effort, we have finally developed the necessary additional techniques. Combining these additional techniques with techniques from previous investigations, we are able to characterize the membership $H_f \in \mathcal{C}_p^-$, $2n < p < \infty$.

It is well known that, if $p, q \in (1, \infty)$ are such that $p^{-1} + q^{-1} = 1$, then $\mathcal{C}_q^+$ is the dual of $\mathcal{C}_p^-$ [9,Section III.15]. This duality was quite useful, sometimes even crucial, in the investigations of many problems in the past. It is, therefore, something of a surprise that this duality plays no role whatsoever in this paper. Instead, we must exploit a different kind of relation between the families $\{\mathcal{C}_p^- : 2 < p < \infty\}$ and $\{\mathcal{C}_p^+ : 2 < p < \infty\}$.

To state our result, it is necessary to recall the notion of *symmetric gauge functions*. Let $\hat{c}$ be the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \neq 0\}$ is finite. A symmetric gauge function (also called *symmetric norming function*) is a map

$$\Phi : \hat{c} \to [0, \infty)$$

that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \ldots, 0, \ldots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [9,page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \ldots, s_j(A), 0, \ldots, 0, \ldots\})$$

for operators. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\}$$

is a norm ideal [9,page 68]. If X is an unbounded operator, then its s-numbers are not defined. But it will be convenient to adopt the convention that $\|X\|_\Phi = \infty$ whenever X is an unbounded operator.

In particular, associated with the ideal $\mathcal{C}_p^-$ is the symmetric gauge function Φ_p^-, which is defined as follows. Let $1 < p < \infty$. For each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, define

$$\Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|a_{\pi(j)}|}{j^{(p-1)/p}},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \geq |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$, which exists because $a_j = 0$ for all but a finite number of j's. Then we have $\mathcal{C}_p^- = \mathcal{C}_{\Phi_p^-}$.

Similarly, for each $1 \leq p < \infty$ we define the symmetric gauge function

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\geq 1} \frac{|a_{\pi(1)}| + |a_{\pi(2)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where, again, $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \geq |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$. Then $\mathcal{C}_p^+ = \mathcal{C}_{\Phi_p^+}$. Theorem 1.6 in [5] implies that if Φ is a symmetric gauge function and if $0 < \|H_f\|_\Phi < \infty$ for some $f \in L^2(S, d\sigma)$, then $\mathcal{C}_\Phi \supset \mathcal{C}_{2n}^+$.

As in [6,18], we need to extend the domains of definition of symmetric gauge functions beyond the space $\hat{c}$. Let Φ be any symmetric gauge function. Suppose that $\{b_j\}_{j\in\mathbf{N}}$ is an arbitrary sequence of real numbers, i.e., suppose that the set $\{j \in \mathbf{N} : b_j \neq 0\}$ is not necessarily finite. Then we define

$$\Phi(\{b_j\}_{j\in\mathbf{N}}) = \sup_{j\geq 1} \Phi(\{b_1, \ldots, b_j, 0, \ldots, 0, \ldots\}). \tag{1.1}$$

Thus for every bounded operator A we can simply write

$$\|A\|_\Phi = \Phi(\{s_1(A), \ldots, s_j(A), \ldots\}).$$

We also need to deal with sequences indexed by sets other than $\mathbf{N}$. If W is a countable, infinite set, then we define

$$\Phi(\{b_\alpha\}_{\alpha\in W}) = \Phi(\{b_{\pi(j)}\}_{j\in\mathbf{N}}),$$

where $\pi : \mathbf{N} \to W$ is any bijection. From the definition of symmetric gauge functions we see that the value of $\Phi(\{b_\alpha\}_{\alpha\in W})$ is independent of the choice of the bijection π. For a finite index set $F = \{x_1, \ldots, x_\ell\}$, we define

$$\Phi(\{b_x\}_{x\in F}) = \Phi(\{b_{x_1}, \ldots, b_{x_\ell}, 0, \ldots, 0, \ldots\}).$$

Let us write $\mathbf{B}$ for the open unit ball $\{z : |z| < 1\}$ in $\mathbf{C}^n$. Let β be the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B},$$

where φ_z is the Möbius transform of $\mathbf{B}$ [15,Section 2.2]. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z, a) = \{w \in \mathbf{B} : \beta(z, w) < a\}$.

Definition 1.1. [18,Definition 1.1] (i) Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z, a) \cap D(w, a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) Let $0 < a < b < \infty$. A subset Γ of $\mathbf{B}$ is said to be an a, b-lattice if it is a-separated and has the property $\cup_{z\in\Gamma} D(z, b) = \mathbf{B}$.

Recall that the normalized reproducing kernel for the Hardy space $H^2(S)$ is given by the formula

$$k_z(w) = \frac{(1 - |z|^2)^{n/2}}{(1 - \langle w, z\rangle)^n}, \quad |z| < 1, \quad |w| \le 1.$$

For $f \in L^2(S, d\sigma)$ and $z \in \mathbf{B}$, we define

$$\mathrm{Var}(f; z) = \|(f - \langle fk_z, k_z\rangle)k_z\|^2.$$

We think of $\mathrm{Var}(f; z)$ as the "variance" of f with respect to the probability measure $|k_z|^2 d\sigma$ on S. We know from previous investigations that the scalar quantity $\mathrm{Var}(f; z)$ plays an extremely important role in the study of Hankel operators.

One can formulate a rather broad conjecture about the membership of Hankel operators H_f in a norm ideal $\mathcal{C}_\Phi$. Suppose that Φ is a symmetric gauge function satisfying the condition $\mathcal{C}_\Phi \supset \mathcal{C}^+_{2n}$, which is necessary for $\mathcal{C}_\Phi$ to contain any $H_f \neq 0$ [5,Theorem 1.6]. Then the general conjecture is that a Hankel operator H_f belongs to $\mathcal{C}_\Phi$ if and only if

$$\Phi(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma}) < \infty$$

for some a, b-lattice Γ in $\mathbf{B}$ with $b \ge 2a$. But the challenge is to prove this conjectured result for specific symmetric gauge functions, where success depends in no small measure on the "user-friendliness" of the Φ in question. In [6], the solution of this problem for the symmetric gauge functions Φ_p^+, $2n < p < \infty$, represented the limit of what could be done with the techniques available then. Now, newly developed techniques allow us to finally solve this problem for the symmetric gauge functions Φ_p^-, $2n < p < \infty$:

Theorem 1.2. *Let $2n < p < \infty$ be given. Let $0 < a < b < \infty$ be positive numbers such that $b \geq 2a$. Then there exist constants $0 < c \leq C < \infty$ which depend only on the given p, a, b and the complex dimension n such that the inequality*

$$c\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma}) \leq \|H_f\|_p^- \leq C\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma})$$

holds for every $f \in L^2(S, d\sigma)$ and every a, b-lattice Γ in $\mathbf{B}$.

Next let us explain some of the difficulties involved in the proof of Theorem 1.2. Recall that in [6], an extremely important role was played by the inequality

$$c\left(\Phi_p^+(\{\alpha_k\}_{k\in\mathbf{N}})\right)^r \leq \Phi_\rho^+(\{\alpha_k^r\}_{k\in\mathbf{N}}) \leq C\left(\Phi_p^+(\{\alpha_k\}_{k\in\mathbf{N}})\right)^r, \tag{1.2}$$

where $1 < r < \infty$, $1 < \rho < \infty$ and $p = \rho r$. For the lack of a better term, one might call (1.2) the *power-transformation property* of the family of symmetric gauge functions Φ_p^+, $1 < p < \infty$. This power-transformation property is needed because, e.g., at certain point in our estimates, what we *can prove* are inequalities of the form

$$\Phi(\{\|A\psi_{z,t}\|^2\}_{z\in F}) = \Phi(\{\langle A^*A\psi_{z,t}, \psi_{z,t}\rangle\}_{z\in F}) \leq C\|A^*A\|_\Phi, \tag{1.3}$$

but what we *need to prove* are inequalities of the form

$$\Phi(\{\|A\psi_{z,t}\|\}_{z\in F}) \leq C\|A\|_\Phi. \tag{1.4}$$

The power-transformation property is precisely what allows us to deduce (1.4) from (1.3). But for this paper, the first stumbling block is that there is no analogue of this power-transformation property for the family of symmetric gauge functions Φ_p^-, $1 < p < \infty$. Thus our only hope is to somehow "make (1.2) work for the Φ_p^--problem", so to speak. Thanks to a rather complicated relation between Φ_p^- and $\Phi_{r'}^+$, Φ_r^+, $1 < r' < p < r < \infty$, this idea actually works.

Another major difficulty is the proof of a "reverse Hölder's inequality" of the form

$$\Phi(\{J_t(g; k, j)\}_{(k,j)\in I}) \leq C\Phi(\{J(g; k, j)\}_{(k,j)\in I}). \tag{1.5}$$

Here $t \geq 1$ and J_t "has the exponent t inside the integral", making (1.5) a reverse Hölder's inequality. In [6], the proof of this inequality in the case of Φ_p^+ again depended on the power-transformation property. But for the proof of this inequality in the case of Φ_p^-, even the above-mentioned relation between Φ_p^- and $\Phi_{r'}^+$, Φ_r^+ does not help. Instead, we must take an entirely new approach. We exploit a property of Φ_p^- called (DQK). Condition (DQK) was introduced in [16] for a completely different purpose, but it turns out to be exactly what is needed to prove (1.5). We are able to show that (1.5) actually holds for every symmetric gauge function that satisfies condition (DQK).

To conclude the Introduction, let us briefly describe the organization of the paper. We begin by establishing the all too important relation between Φ_p^- and $\Phi_{r'}^+$, Φ_r^+ in Section 2. Using the results from Section 2 and the *partial sampling* technique from [6], in Section 3 we prove (1.4) in the case $\Phi = \Phi_p^-$, $2 < p < \infty$, which is one of the two main steps in the proof of the lower bound in Theorem 1.2. Section 4 is the other main step in the proof of the lower bound, which involves the local inequality from [6]. The proof of the lower bound is then completed in Section 5. Section 6 is devoted to the reverse Hölder's inequality mentioned above. Finally, using the inequalities from Sections 2 and 6, in Section 7 we prove the upper bound in Theorem 1.2 through a two-stage interpolation.

2. Symmetric gauge functions Φ_p^- and Φ_r^+

The proof of Theorem 1.2 depends on a crucial relation between the symmetric gauge functions Φ_p^- and Φ_r^+, $\Phi_{r'}^+$, where $1 < r' < p < r < \infty$. Our task in this section is to establish this relation.

Let us introduce the following notation. For every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$ and every $s > 0$, we denote

$$N(a; s) = \mathrm{card}\{j \in \mathbf{N} : a_j > s\}.$$

Lemma 2.1. *Let $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$ we have*

$$\int_0^\infty \{N(a; s)\}^{1/p} ds \le \Phi_p^-(a) \le p \int_0^\infty \{N(a; s)\}^{1/p} ds. \tag{2.1}$$

Proof. Given any $1 < p < \infty$, it is trivial that

$$k^{1/p} \le \sum_{j=1}^k \frac{1}{j^{(p-1)/p}} \le 1 + \int_1^k \frac{1}{x^{(p-1)/p}} dx \le p k^{1/p} \tag{2.2}$$

for every $k \in \mathbf{N}$. For the given p, define the measure μ_p^- on $\mathbf{N}$ by the formula

$$\mu_p^-(E) = \sum_{j \in E} \frac{1}{j^{(p-1)/p}}, \quad E \subset \mathbf{N}.$$

By the monotone convergence theorem and (1.1), it suffices to consider the case where the sequence $a = \{a_1, \dots, a_j, \dots\}$ has only a finite number of nonzero terms. For such a sequence, rearranging the terms if necessary, we may assume that it is non-increasing, i.e.,

$$a_1 \ge a_2 \ge \cdots \ge a_j \ge \cdots .$$

For such an $a = \{a_1, \ldots, a_j, \ldots\}$ we have

$$(2.3) \qquad \Phi_p^-(a) = \sum_{j=1}^{\infty} \frac{a_j}{j^{(p-1)/p}} = \int_0^\infty \mu_p^-(\{j \in \mathbf{N} : a_j > s\})ds,$$

where the second = follows from Fubini's theorem. Suppose that $a_1 > 0$, for otherwise (2.1) holds trivially. Since the sequence $a = \{a_1, \ldots, a_j, \ldots\}$ is non-increasing, for each $0 < s < a_1$ we have $a_j > s$ if $1 \le j \le N(a; s)$ and $a_j \le s$ if $j > N(a; s)$. Thus for every $0 < s < a_1$ we have

$$\mu_p^-(\{j \in \mathbf{N} : a_j > s\}) = \mu_p^-(\{1, \ldots, N(a; s)\}) = \sum_{j=1}^{N(a;s)} \frac{1}{j^{(p-1)/p}}.$$

Combining this with (2.2), we obtain

$$(2.4) \qquad \{N(a; s)\}^{1/p} \le \mu_p^-(\{j \in \mathbf{N} : a_j > s\}) \le p\{N(a; s)\}^{1/p}$$

for $0 < s < a_1$. On the other hand, it is obvious that if $s \ge a_1$, then

$$(2.5) \qquad \mu_p^-(\{j \in \mathbf{N} : a_j > s\}) = \mu_p^-(\emptyset) = 0 = \{N(a; s)\}^{1/p}.$$

Obviously, (2.1) follows from the combination of (2.3), (2.4) and (2.5). □

Proposition 2.2. *For every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$ and every $s > 0$, define the sequence $a^\vee(s) = \{a_1^\vee(s), \ldots, a_j^\vee(s), \ldots\}$, where*

$$a_j^\vee(s) = \begin{cases} 0 & \text{if} \quad a_j > s \\ a_j & \text{if} \quad a_j \le s \end{cases}, \quad j \in \mathbf{N}.$$

Then given any $1 < p < r < \infty$, there exists a constant $0 < C_{2.2} < \infty$ such that

$$(2.6) \qquad \int_0^\infty \left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} ds \le C_{2.2}\Phi_p^-(a)$$

for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$.

Proof. Let $1 < p < r < \infty$ be given. By the monotone convergence theorem and (1.1), it suffices to consider the case where $a = \{a_1, \ldots, a_j, \ldots\}$ has only a finite number of nonzero terms. For each $i \in \mathbf{Z}$, define

$$(2.7) \qquad \nu(i) = \text{card}\{j \in \mathbf{N} : 2^{-i} < a_j \le 2^{-i+1}\}.$$

Suppose that $2^{-i} < s \le 2^{-i+1}$ for some $i \in \mathbf{Z}$. For such an s, by the definition of Φ_r^+, there is a subset $\mathcal{E}(s)$ of $\mathbf{N}$ with $\text{card}(\mathcal{E}(s)) = k(s) \in \mathbf{N}$ such that

$$\Phi_r^+(a^\vee(s)) = \frac{\sum_{j \in \mathcal{E}(s)} a_j^\vee(s)}{1^{-1/r} + \cdots + (k(s))^{-1/r}} \le \frac{\sum_{j \in \mathcal{E}(s)} a_j^\vee(s)}{(k(s))^{1-(1/r)}}.$$

Define $E_{s,m} = \{j \in \mathcal{E}(s) : 2^{-i-m} < a_j^\vee(s) \le 2^{-i-m+1}\}$, $m \in \mathbf{Z}_+$. Since $a_j^\vee(s) \le s$ for every j and since $s \le 2^{-i+1}$, we have

$$\sum_{j\in\mathcal{E}(s)} a_j^\vee(s) = \sum_{m=0}^{\infty} \sum_{j\in E_{s,m}} a_j^\vee(s).$$

If j, i and m are such that $a_j^\vee(s) > 2^{-i-m}$, then $a_j^\vee(s) = a_j$. Therefore

$$\mathrm{card}(E_{s,m}) \le \min\{\nu(i+m), k(s)\}.$$

Hence for each $m \ge 0$ we have

$$\frac{1}{(k(s))^{1-(1/r)}} \sum_{j\in E_{s,m}} a_j^\vee(s) \le \frac{2}{2^{i+m}} \cdot \frac{\mathrm{card}(E_{s,m})}{(k(s))^{1-(1/r)}} \le \frac{2}{2^{i+m}} \{\nu(i+m)\}^{1/r}.$$

Combining this with the above, we conclude that if $2^{-i} < s \le 2^{-i+1}$, then

$$\Phi_r^+(a^\vee(s)) \le \frac{1}{(k(s))^{1-(1/r)}} \sum_{m=0}^{\infty} \sum_{j\in E_{s,m}} a_j^\vee(s) \le 2 \sum_{m=0}^{\infty} \frac{1}{2^{i+m}} \{\nu(i+m)\}^{1/r}.$$

Consequently, we have

$$\frac{1}{s}\Phi_r^+(a^\vee(s)) \le 2 \sum_{m=0}^{\infty} \frac{1}{2^m} \{\nu(i+m)\}^{1/r} \quad \text{for every} \quad s \in (2^{-i}, 2^{-i+1}]. \tag{2.8}$$

Since $r/p > 1$, we have $r/p = (1+\epsilon)/(1-\epsilon)$ for some $0 < \epsilon < 1$. That is, $(r/p)(1-\epsilon) = 1+\epsilon$. Factoring 2^{-m} in the form $2^{-m} = 2^{-\epsilon m} \cdot 2^{-(1-\epsilon)m}$, a simple application of Hölder's inequality to (2.8) gives us

$$\left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} \le C \sum_{m=0}^{\infty} \frac{1}{2^{(1+\epsilon)m}} \{\nu(i+m)\}^{1/p}$$

for $s \in (2^{-i}, 2^{-i+1}]$. Therefore

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} ds &= \sum_{i=-\infty}^{\infty} \int_{2^{-i}}^{2^{-i+1}} \left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} ds \\
&\le C \sum_{i=-\infty}^{\infty} \frac{1}{2^i} \sum_{m=0}^{\infty} \frac{1}{2^{(1+\epsilon)m}} \{\nu(i+m)\}^{1/p} = C \sum_{i=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2^{\epsilon m}} \cdot \frac{1}{2^{i+m}} \{\nu(i+m)\}^{1/p} \\
&= C \sum_{k=-\infty}^{\infty} \frac{1}{2^k} \{\nu(k)\}^{1/p} \sum_{m=0}^{\infty} \frac{1}{2^{\epsilon m}} = C_1 \sum_{k=-\infty}^{\infty} \frac{1}{2^k} \{\nu(k)\}^{1/p}.
\end{aligned} \tag{2.9}$$

By (2.7), we have $\nu(k) \le N(a;s)$ for every $s \in (2^{-k-1}, 2^{-k}]$. Thus

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{1}{2^k}\{\nu(k)\}^{1/p} &= 2\sum_{k=-\infty}^{\infty} \frac{1}{2^{k+1}}\{\nu(k)\}^{1/p} \le 2\sum_{k=-\infty}^{\infty}\int_{2^{-k-1}}^{2^{-k}}\{N(a;s)\}^{1/p}ds\\ &= 2\int_0^\infty \{N(a;s)\}^{1/p}ds \le 2\Phi_p^-(a),\end{aligned} \tag{2.10}$$

where the last $\le$ is an application of Lemma 2.1. Obviously, the proposition follows from the combination of (2.9) and (2.10). □

Proposition 2.3. *For every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$ and every $s > 0$, define the sequence $a^\wedge(s) = \{a_1^\wedge(s), \ldots, a_j^\wedge(s), \ldots\}$, where*

$$a_j^\wedge(s) = \begin{cases} a_j & \textit{if} \quad a_j > s \\ 0 & \textit{if} \quad a_j \le s \end{cases}, \quad j \in \mathbf{N}.$$

Then given any $1 < r' < p < \infty$, there exists a constant $0 < C_{2.3} < \infty$ such that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r'}^+(a^\wedge(s))\right)^{r'/p} ds \le C_{2.3}\Phi_p^-(a)$$

for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$.

Proof. Let $1 < r' < p < \infty$ be given. Again, by the monotone convergence theorem and (1.1), it suffices to consider the case where $a = \{a_1, \ldots, a_j, \ldots\}$ has only a finite number of nonzero terms. For each $i \in \mathbf{Z}$, let $\nu(i)$ be given by (2.7). Suppose that $2^{-i} < s \le 2^{-i+1}$ for some $i \in \mathbf{Z}$. By the definition of $\Phi_{r'}^+$, there is a subset $\mathcal{F}(s)$ of $\mathbf{N}$ with $\mathrm{card}(\mathcal{F}(s)) = k'(s) \in \mathbf{N}$ such that

$$\Phi_{r'}^+(a^\wedge(s)) = \frac{\sum_{j\in\mathcal{F}(s)} a_j^\wedge(s)}{1^{-1/r'} + \cdots + (k'(s))^{-1/r'}} \le \frac{\sum_{j\in\mathcal{F}(s)} a_j^\wedge(s)}{(k'(s))^{1-(1/r')}}.$$

Define $F_{s,m} = \{j \in \mathcal{F}(s) : 2^{-i+m} < a_j^\wedge(s) \le 2^{-i+m+1}\}$ for each $m \in \mathbf{Z}_+$. By definition, if $a_j^\wedge(s) > 0$, then $a_j^\wedge(s) > s$. Since $s > 2^{-i}$, we have

$$\sum_{j\in\mathcal{F}(s)} a_j^\wedge(s) = \sum_{m=0}^{\infty}\sum_{j\in F_{s,m}} a_j^\wedge(s).$$

We have $\mathrm{card}(F_{s,m}) \le \min\{\nu(i-m), k'(s)\}$ for every $m \ge 0$. Therefore

$$\Phi_{r'}^+(a^\wedge(s)) \le \frac{1}{(k'(s))^{1-(1/r')}}\sum_{m=0}^{\infty}\sum_{j\in F_{s,m}} a_j^\wedge(s) \le 2\sum_{m=0}^{\infty} 2^{-i+m}\{\nu(i-m)\}^{1/r'}.$$

Consequently,

$$\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s)) \le 2\sum_{m=0}^{\infty} 2^{m}\{\nu(i-m)\}^{1/r'}.$$

Since $0 < r'/p < 1$, it follows that

$$\left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/p} \le 2\sum_{m=0}^{\infty} 2^{mr'/p}\{\nu(i-m)\}^{1/p}$$

for $2^{-i} < s \le 2^{-i+1}$. Thus

$$\begin{aligned}
\int_0^{\infty}\left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/p} ds &= \sum_{i=-\infty}^{\infty}\int_{2^{-i}}^{2^{-i+1}}\left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/p} ds \\
&\le 2\sum_{i=-\infty}^{\infty} 2^{-i}\sum_{m=0}^{\infty} 2^{mr'/p}\{\nu(i-m)\}^{1/p} \\
&= 2\sum_{i=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2^{(1-(r'/p))m}}\cdot\frac{1}{2^{i-m}}\{\nu(i-m)\}^{1/p} \\
&= 2\sum_{k=-\infty}^{\infty}\frac{1}{2^{k}}\{\nu(k)\}^{1/p}\sum_{m=0}^{\infty}\frac{1}{2^{(1-(r'/p))m}}.
\end{aligned}$$

Recalling (2.10), the proof is now complete. □

Although Theorem 1.2 is about membership in the ideal $\mathcal{C}_p^-$, the fact that we need Propositions 2.2 and 2.3 clearly indicates that symmetric gauge functions Φ_p^+, $1 < p < \infty$, will be an important part of our analysis. We end this section with some facts about these symmetric gauge functions, which will be needed later on.

Lemma 2.4. [6,Lemma 5.6] *Suppose that $1 < p < \infty$. Let $\alpha = \{\alpha_1, \dots, \alpha_k, \dots\}$ be a non-increasing sequence of non-negative numbers. Define*

$$F_p(\alpha) = \sup_{k\ge 1} k^{1/p}\alpha_k.$$

Then

$$\frac{p-1}{p}F_p(\alpha) \le \Phi_p^+(\alpha) \le F_p(\alpha).$$

Lemma 2.5. [6,Lemma 5.7] *Let $1 < r < \infty$, $1 < \rho < \infty$ and $p = \rho r$. Then for every sequence $\alpha = \{\alpha_1, \dots, \alpha_k, \dots\}$ of non-negative numbers we have*

$$\frac{\rho-1}{\rho}\left(\Phi_p^+(\{\alpha_k\}_{k\in\mathbf{N}})\right)^r \le \Phi_\rho^+(\{\alpha_k^r\}_{k\in\mathbf{N}}) \le \left(\frac{p}{p-1}\Phi_p^+(\{\alpha_k\}_{k\in\mathbf{N}})\right)^r.$$

If Φ_p denotes the symmetric gauge function for the Schatten class $\mathcal{C}_p$, $1 < p < \infty$, then, of course, for every sequence of non-negative numbers $a = \{a_1, \dots, a_k, \dots\}$ we have the following well-known inequality of weak-type:

$$N(a; s) \le (\Phi_p(a)/s)^p \tag{2.11}$$

for $s > 0$. But for the purpose of this paper, (2.11) is not good enough; we need an improved version of it. More specifically, we need to replace the $\Phi_p(a)$ above by $\Phi_p^+(a)$:

Lemma 2.6. *Suppose that $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \dots, a_k, \dots\}$ and every $s > 0$ we have*

$$N(a; s) \le \left(\frac{p}{p-1}\right)^p \left(\frac{1}{s}\Phi_p^+(a)\right)^p. \tag{2.12}$$

Proof. Given an $s > 0$, set $M = \{j \in \mathbf{N} : a_j > s\}$. If $\mathrm{card}(M) = \infty$, then $\Phi_p^+(a) = \infty$, and therefore (2.12) holds in this case. Obviously, (2.12) also holds in the case $M = \emptyset$. Suppose that $\mathrm{card}(M) = m \in \mathbf{N}$. Then there is a bijection $\pi : \{1, \dots, m\} \to M$ such that

$$a_{\pi(1)} \ge \cdots \ge a_{\pi(m)}.$$

Since $a_{\pi(m)} > s$, by Lemma 2.4 we have

$$sm^{1/p} < a_{\pi(m)} m^{1/p} \le \sup_{1 \le k \le m} a_{\pi(k)} k^{1/p} \le \frac{p}{p-1}\Phi_p^+(\{a_{\pi(1)}, \cdots, a_{\pi(m)}\}) \le \frac{p}{p-1}\Phi_p^+(a).$$

Solving for m $(= N(a; s))$, we find that $m \le \{p/(p-1)\}^p (\Phi_p^+(a)/s)^p$. $\square$

Although (2.12) is only a slight improvement of (2.11), we will see in Sections 3 and 7 that this improvement makes quite a difference. In fact, (2.12) is the reason why Propositions 2.2 and 2.3 are useful for our purpose.

3. Decomposition and modified kernel

It is well known that the formula

$$d(\zeta, \xi) = |1 - \langle \zeta, \xi \rangle|^{1/2}, \qquad \zeta, \xi \in S, \tag{3.1}$$

defines a metric on S [15,page 66]. Throughout the paper, we denote

$$B(\zeta, r) = \{x \in S : |1 - \langle x, \zeta \rangle|^{1/2} < r\}$$

for $\zeta \in S$ and $r > 0$. There is a constant $2^{-n} < A_0 < \infty$ such that

$$2^{-n} r^{2n} \le \sigma(B(\zeta, r)) \le A_0 r^{2n} \tag{3.2}$$

for all $\zeta \in S$ and $0 < r \le \sqrt{2}$ [15,Proposition 5.1.4]. Note that the upper bound actually holds when $r > \sqrt{2}$.

Next we need to recall the spherical decomposition in [6]. For each integer $k \geq 0$, let $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ be a subset of S which is *maximal* with respect to the property

$$B(u_{k,j}, 2^{-k-1}) \cap B(u_{k,j'}, 2^{-k-1}) = \emptyset \quad \text{for all} \quad 1 \leq j < j' \leq m(k). \tag{3.3}$$

The maximality of $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ implies that

$$\cup_{j=1}^{m(k)} B(u_{k,j}, 2^{-k}) = S. \tag{3.4}$$

For each pair of $k \geq 0$ and $1 \leq j \leq m(k)$, define

$$T_{k,j} = \{ru : 1 - 2^{-2k} \leq r^2 < 1 - 2^{-2(k+1)}, u \in B(u_{k,j}, 2^{-k})\}. \tag{3.5}$$

As in [6], we define the index set

$$I = \{(k, j) : k \geq 0, 1 \leq j \leq m(k)\}.$$

Recall from [5,6] that for each pair of $0 < t < \infty$ and $z \in \mathbf{B}$, we define

$$\psi_{z,t}(\zeta) = \frac{(1 - |z|^2)^{(n/2)+t}}{(1 - \langle \zeta, z \rangle)^{n+t}},$$

$|\zeta| \leq 1$. In terms of the normalized reproducing kernel k_z and the Schur multiplier

$$m_z(\zeta) = \frac{1 - |z|}{1 - \langle \zeta, z \rangle}, \tag{3.6}$$

we have the relation

$$\psi_{z,t} = (1 + |z|)^t m_z^t k_z.$$

We think of $\psi_{z,t}$ as a modified kernel function, i.e., a modified version of k_z.

Definition 3.1. [6,Definition 3.2] (a) A *partial sampling* set is a finite subset F of the open unit ball $\mathbf{B}$ with the property that $\text{card}(F \cap T_{k,j}) \leq 1$ for every $(k, j) \in I$.
(b) For any partial sampling set F and any $t > 0$, denote

$$R_F^{(t)} = \sum_{z \in F} \psi_{z,t} \otimes \psi_{z,t}.$$

The next proposition shows the benefit of modifying k_z:

Proposition 3.2. *For each $t > 0$, there is a constant $C_{3.2}(t)$ such that the inequality*

$$\Phi(\{\langle B\psi_{z,t}, \psi_{z,t} \rangle\}_{z \in F}) \leq C_{3.2}(t)\|B\|_\Phi$$

holds for every partial sampling set F, every symmetric gauge function Φ, and every non-negative self-adjoint operator B on the Hardy space $H^2(S)$.

Proof. Let Φ be any symmetric gauge function. Then it has the following property: For non-negative numbers $a_1 \geq \cdots \geq a_\nu \geq 0$ and $b_1 \geq \cdots \geq b_\nu \geq 0$ in descending order, if

$$a_1 + \cdots + a_j \leq b_1 + \cdots + b_j \quad \text{for every} \quad 1 \leq j \leq \nu,$$

then

$$\Phi(\{a_1, \ldots, a_\nu, 0, \ldots, 0 \ldots\}) \leq \Phi(\{b_1, \ldots, b_\nu, 0, \ldots, 0 \ldots\}).$$

See Lemma III.3.1 in [9]. Let $t > 0$ be given. By [6,Proposition 3.3], there is a constant $C_{3.2}(t)$ such that

$$\|R_F^{(t)}\| \leq C_{3.2}(t)$$

for every partial sampling set F.

Let B be a non-negative self-adjoint operator, and suppose that F is a partial sampling set with $\mathrm{card}(F) = m$. Then we can enumerate the elements in F as $z_1, \ldots, z_m$ in such a way that

$$\langle B\psi_{z_1,t}, \psi_{z_1,t}\rangle \geq \cdots \geq \langle B\psi_{z_m,t}, \psi_{z_m,t}\rangle.$$

For each $1 \leq k \leq m$, define the subset $F_k = \{z_1, \ldots, z_k\}$ of F. Then each F_k is also a partial sampling set, and we have $\|R_{F_k}^{(t)}\| \leq \|R_F^{(t)}\| \leq C_{3.2}(t)$ for every $1 \leq k \leq m$. In terms of s-numbers, this implies that

$$s_j(BR_{F_k}^{(t)}) \leq C_{3.2}(t)s_j(B)$$

for every $j \geq 1$ (see page 61 in [9]). Write $\|\cdot\|_1$ for the norm of the trace class. Since $\mathrm{rank}(R_{F_k}^{(t)}) \leq k$, we have

$$\begin{aligned}\langle B\psi_{z_1,t}, \psi_{z_1,t}\rangle + \cdots + \langle B\psi_{z_k,t}, \psi_{z_k,t}\rangle &= \mathrm{tr}(BR_{F_k}^{(t)}) \leq \|BR_{F_k}^{(t)}\|_1 \\ &= s_1(BR_{F_k}^{(t)}) + \cdots + s_k(BR_{F_k}^{(t)}) \leq C_{3.2}(t)\{s_1(B) + \cdots + s_k(B)\}.\end{aligned}$$

Since this holds for every $1 \leq k \leq m$, by the property of Φ that we mentioned in the previous paragraph, we have

$$\Phi(\{\langle B\psi_{z,t}, \psi_{z,t}\rangle\}_{z\in F}) \leq C_{3.2}(t)\Phi(\{s_j(B)\}_{j\in\mathbf{N}}) = C_{3.2}(t)\|B\|_\Phi,$$

proving the proposition. □

Proposition 3.3. *Given any pair of $t > 0$ and $2 < p < \infty$, there exists a constant $C_{3.3}(t,p)$ such that the inequality*

$$\Phi_p^+(\{\|A\psi_{z,t}\|\}_{z\in F}) \leq C_{3.3}(t,p)\|A\|_p^+$$

holds for every bounded operator $A : H^2(S) \to L^2(S, d\sigma)$ and every partial sampling set F.

Proof. Let $t > 0$ and $2 < p < \infty$ be given. Set $\rho = p/2$. Then $\rho > 1$ and $p = 2\rho$. Let $C = \{\rho/(\rho-1)\}^{1/2}$. Let $A : H^2(S) \to L^2(S, d\sigma)$ be any bounded operator and let F be any partial sampling set. Applying Lemma 2.5 with $r = 2$, we have

$$\begin{aligned} \Phi_p^+(\{\|A\psi_{z,t}\|\}_{z\in F}) &\le C\left(\Phi_\rho^+(\{\|A\psi_{z,t}\|^2\}_{z\in F})\right)^{1/2} \\ &= C\left(\Phi_\rho^+(\{\langle A^*A\psi_{z,t}, \psi_{z,t}\rangle\}_{z\in F})\right)^{1/2}. \end{aligned} \tag{3.7}$$

On the other hand, Proposition 3.2 gives us

$$\Phi_\rho^+(\{\langle A^*A\psi_{z,t}, \psi_{z,t}\rangle\}_{z\in F}) \le C_{3.2}(t)\|A^*A\|_\rho^+. \tag{3.8}$$

Again applying Lemma 2.5 with $r = 2$, we have

$$\|A^*A\|_\rho^+ = \|(A^*A)^{2/2}\|_\rho^+ \le \left\{\frac{p}{p-1}\|(A^*A)^{1/2}\|_p^+\right\}^2 = \left\{\frac{p}{p-1}\|A\|_p^+\right\}^2. \tag{3.9}$$

Thus if we set $C_{3.3}(t,p) = C\{C_{3.2}(t)\}^{1/2}\{p/(p-1)\}$, then the proposition follows from the combination of (3.7), (3.8) and (3.9). $\square$

Proposition 3.4. *Given any pair of $t > 0$ and $2 < p < \infty$, there exists a constant $C_{3.4}(t,p)$ such that the inequality*

$$\Phi_p^-(\{\|A\psi_{z,t}\|\}_{z\in F}) \le C_{3.4}(t,p)\|A\|_p^- \tag{3.10}$$

holds for every bounded operator $A : H^2(S) \to L^2(S, d\sigma)$ and every partial sampling set F.

Proof. Let $t > 0$ and $2 < p < \infty$ be given. We pick an r' such that $2 < r' < p$. To prove (3.10), we only need to consider compact $A : H^2(S) \to L^2(S, d\sigma)$, for otherwise the inequality holds for the trivial reason that its right-hand side is infinity. But for a compact A, we have the representation

$$A = \sum_{j=1}^\infty a_j x_j \otimes y_j,$$

where $\{x_j : j \in \mathbf{N}\}$ and $\{y_j : j \in \mathbf{N}\}$ are orthonormal sets in $L^2(S, d\sigma)$ and $H^2(S)$ respectively, and $a_j \ge 0$ for every $j \in \mathbf{N}$. For every $s > 0$, define the operators

$$A_s = \sum_{a_j > s} a_j x_j \otimes y_j \quad \text{and} \quad B_s = \sum_{a_j \le s} a_j x_j \otimes y_j.$$

It follows from Proposition 2.3 that

$$\int_0^\infty \left(\frac{1}{s}\|A_s\|_{r'}^+\right)^{r'/p} ds \le C_{2.3}\|A\|_p^-. \tag{3.11}$$

On the other hand, it is obvious that $\|B_s\| \le s$. Since $\|\psi_{z,t}\| \le 2^t$, we have

$$\|B_s\psi_{z,t}\| \le 2^t s \tag{3.12}$$

for all $z \in \mathbf{B}$ and $s > 0$.

Let a partial sampling set F be given. With somewhat abuse of notation, let us write

$$N(F;\lambda) = \text{card}\{z \in F : \|A\psi_{z,t}\| > \lambda\}$$

for $\lambda > 0$. By Lemma 2.1, we have

$$\text{(3.13)}\quad \Phi_p^-(\{\|A\psi_{z,t}\|\}_{z\in F}) \le p\int_0^\infty \{N(F;\lambda)\}^{1/p}d\lambda = (1+2^t)p\int_0^\infty \{N(F;(1+2^t)s)\}^{1/p}ds,$$

where the last step is the substitution $\lambda = (1+2^t)s$. Define

$$N(s) = \text{card}\{z \in F : \|A_s\psi_{z,t}\| > s\}$$

for $s > 0$. Since $A = A_s + B_s$, we have $\|A\psi_{z,t}\| \le \|A_s\psi_{z,t}\| + \|B_s\psi_{z,t}\|$ for all $s > 0$ and $z \in F$. Therefore (3.12) implies that for every $s > 0$,

$$N(F;(1+2^t)s) \le N(s).$$

Applying Lemma 2.6 and Proposition 3.3, we have

$$N(s) \le \left(\frac{r'}{r'-1}\right)^{r'}\left(\frac{1}{s}\Phi_{r'}^+(\{\|A_s\psi_{z,t}\|\}_{z\in F})\right)^{r'} \le \left(\frac{r'}{r'-1}\right)^{r'}\left(\frac{1}{s}C_{3.3}(t,r')\|A_s\|_{r'}^+\right)^{r'}.$$

Thus if we set $C = \{r'C_{3.3}(t,r')/(r'-1)\}^{r'/p}$, then

$$\{N(F;(1+2^t)s)\}^{1/p} \le \{N(s)\}^{1/p} \le C\left(\frac{1}{s}\|A_s\|_{r'}^+\right)^{r'/p}$$

for every $s > 0$. Substituting this in (3.13) and recalling (3.11), we obtain

$$\Phi_p^-(\{\|A\psi_{z,t}\|\}_{z\in F}) \le (1+2^t)pC\int_0^\infty \left(\frac{1}{s}\|A_s\|_{r'}^+\right)^{r'/p} ds \le (1+2^t)pCC_{2.3}\|A\|_p^-.$$

This completes the proof of the proposition. □

Definition 3.5. A *partial sampling map* is a map φ from a set X into $\mathbf{B}$ which has the property that $\text{card}\{x \in X : \varphi(x) \in T_{k,j}\} \le 1$ for every $(k,j) \in I$.

Lemma 3.6. *There exists a natural number $M_{3.6}$ determined by the complex dimension n such that the following is true: Let L be a subset of I and suppose that $z : L \to \mathbf{B}$ is a map satisfying the condition $z(k,j) \in T_{k,j}$ for every $(k,j) \in L$. Then there is a partition*

$$L = E_1 \cup \cdots \cup E_{M_{3.6}}$$

such that for every $1 \le \nu \le M_{3.6}$, the map $z : E_\nu \to \mathbf{B}$ is a partial sampling map.

Proof. By (3.5), we have $T_{k,j} \cap T_{k',i} = \emptyset$ for all $k \neq k'$ in $\mathbf{Z}_+$ and $1 \leq j \leq m(k)$, $1 \leq i \leq m(k')$. By (3.2), (3.3) and (3.5), there is an $M \in \mathbf{N}$ determined by the complex dimension n such that the inequality

$$\text{card}\{i : 1 \leq i \leq m(k),\ T_{k,j} \cap T_{k,i} \neq \emptyset\} \leq M \tag{3.14}$$

holds for every $(k, j) \in I$. Let us show that $M_{3.6} = M^2$ suffices for our purpose.

Let $L \subset I$, and suppose that $z : L \to \mathbf{B}$ is a map such that $z(k, j) \in T_{k,j}$ for every $(k, j) \in L$. Then by (3.14), for every $(k, j) \in I$ we have

$$\sum_{T_{k,j} \cap T_{k,i} \neq \emptyset} \text{card}\{\ell \in \{1, \ldots, m(k)\} : z(k, \ell) \in T_{k,i}\} \leq M^2 = M_{3.6}. \tag{3.15}$$

We pick a subset E_1 of L that is maximal with respect to the condition that the restricted map $z : E_1 \to \mathbf{B}$ be a partial sampling map. Suppose that $m \geq 1$ and that we have defined pairwise disjoint subsets $E_1, \ldots, E_m$ of L. We then define E_{m+1} to be a subset of $L \backslash (E_1 \cup \cdots \cup E_m)$ that is maximal with respect to the condition that the restricted map $z : E_{m+1} \to \mathbf{B}$ be a partial sampling map. Then the proof will be complete once we show that $E_{M_{3.6}+1} = \emptyset$. Assume the contrary, i.e., assume that there were some $(k^*, j^*) \in E_{M_{3.6}+1}$. We will show that this leads to a contradiction.

First of all, we have

$$z(k^*, j^*) \in T_{k^*,j^*}. \tag{3.16}$$

By the maximality of the sets $E_1, \ldots, E_{M_{3.6}}$, for each $1 \leq \nu \leq M_{3.6}$, the map z fails to satisfy Definition 3.5 on the set $E_\nu \cup \{(k^*, j^*)\}$. Since z is partial sampling on E_ν, this means that for each $1 \leq \nu \leq M_{3.6}$ there is a $(k_\nu, \ell_\nu) \in E_\nu$ such that

$$\{z(k_\nu, \ell_\nu), z(k^*, j^*)\} \subset T_{k'_\nu, i_\nu}$$

for some $(k'_\nu, i_\nu) \in I$. By (3.16), this implies $k'_\nu = k^* = k_\nu$ and $T_{k^*,i_\nu} \cap T_{k^*,j^*} \neq \emptyset$ for every $1 \leq \nu \leq M_{3.6}$. Thus z maps the set $\{(k^*, j^*), (k^*, \ell_1), \ldots, (k^*, \ell_{M_{3.6}})\}$ into

$$\bigcup_{T_{k^*,j^*} \cap T_{k^*,i} \neq \emptyset} T_{k^*,i}.$$

Since the set $\{(k^*, j^*), (k^*, \ell_1), \ldots, (k^*, \ell_{M_{3.6}})\}$ contains $M_{3.6} + 1 = M^2 + 1$ elements, this contradicts (3.15). This completes the proof of the lemma. □

In addition to the index set I, let us also define $I_m = \{(k, j) \in I : k \leq m\}$ for each $m \in \mathbf{Z}_+$. The following is the main goal of this section:

Proposition 3.7. *Let $2 < p < \infty$ and $0 < t < \infty$. Suppose that $w_{k,j} \in T_{k,j}$ for every $(k, j) \in I$. Then the inequality*

$$\Phi_p^-(\{\|A\psi_{w_{k,j},t}\|\}_{(k,j) \in I_m}) \leq C_{3.4}(t, p) M_{3.6} \|A\|_p^- \tag{3.17}$$

holds for every bounded operator $A : H^2(S) \to L^2(S, d\sigma)$ *and every* $m \geq 1$, *where* $C_{3.4}(t, p)$ *and* $M_{3.6}$ *are the constants provided by Proposition* 3.4 *and Lemma* 3.6 *respectively.*

Proof. First of all, a symmetric gauge function Φ has the following obvious property: If X is any countable set and if $X = X_1 \cup \cdots \cup X_N$, then for every map $\varphi : X \to [0, \infty)$ we have

$$\Phi(\{\varphi(x)\}_{x \in X}) \leq \Phi(\{\varphi(x)\}_{x \in X_1}) + \cdots + \Phi(\{\varphi(x)\}_{x \in X_N}). \tag{3.18}$$

Let $m \geq 1$ be given and consider the map $(k, j) \mapsto w_{k,j}$ from I_m into $\mathbf{B}$. Since $w_{k,j} \in T_{k,j}$ for every (k, j), by Lemma 3.6 there is a partition

$$I_m = E_1 \cup \cdots \cup E_{M_{3.6}}$$

such that for every $1 \leq i \leq M_{3.6}$, the map $(k, j) \mapsto w_{k,j}$ is partial sampling on E_i. By Definition 3.5, this means that the map $(k, j) \mapsto w_{k,j}$ is injective on E_i and $\{w_{k,j} : (k, j) \in E_i\}$ is a partial sampling set as defined in Definition 3.1. Hence Proposition 3.4 gives us

$$\Phi_p^-(\{\|A\psi_{w_{k,j},t}\|\}_{(k,j) \in E_i}) \leq C_{3.4}(t, p)\|A\|_p^-$$

for every bounded operator $A : H^2(S) \to L^2(S, d\sigma)$ and every $1 \leq i \leq M_{3.6}$. By (3.18), we also have

$$\Phi_p^-(\{\|A\psi_{w_{k,j},t}\|\}_{(k,j) \in I_m}) \leq \sum_{i=1}^{M_{3.6}} \Phi_p^-(\{\|A\psi_{w_{k,j},t}\|\}_{(k,j) \in E_i}).$$

Obviously, the proposition follows from the above two inequalities. $\square$

4. Radial contractions and local inequality

As in [6], for each $\ell \in \mathbf{N}$ we define the radial contraction

$$\rho_\ell(z) = \begin{cases} (1 - 4^\ell(1 - |z|^2))^{1/2}(z/|z|) & \text{if} \quad 4^\ell(1 - |z|^2) < 1 \\ \\ 0 & \text{if} \quad 4^\ell(1 - |z|^2) \geq 1 \end{cases}, \tag{4.1}$$

$z \in \mathbf{B}$. One can better understand these ρ_ℓ in terms of the following relations: we have

$$\begin{cases} \rho_\ell(z)/|\rho_\ell(z)| = z/|z| \qquad \text{and} \\ \\ 1 - |\rho_\ell(z)|^2 = 4^\ell(1 - |z|^2) \end{cases} \tag{4.2}$$

if $4^\ell(1 - |z|^2) < 1$. Recall that the Schur multiplier m_z is given by (3.6). A key ingredient in the proof of the lower bound in Theorem 1.2 is the following local inequality for Hankel operators:

Theorem 4.1. [6,Theorem 1.1] *Given any* $0 < \delta \leq 1/2$, *there exists a constant* $0 < C(\delta) < \infty$ *which depends only on* δ *and the complex dimension* n *such that the inequality*

$$\mathrm{Var}^{1/2}(f - Pf; z) \leq C(\delta) \sum_{\ell=1}^{\infty} \frac{1}{2^{(1-\delta)\ell}} \|M_{m_{\rho_\ell(z)}} H_f k_{\rho_\ell(z)}\|$$

holds for all $f \in L^2(S, d\sigma)$ *and* $z \in \mathbf{B}$.

Next we again turn to the symmetric gauge function Φ_p^-.

Lemma 4.2. [6,Lemma 6.1] *Let* $1 < p < \infty$. *Let* X, Y *be countable sets and let* $N \in \mathbf{N}$. *Suppose that* $T : X \to Y$ *is a map that is at most* N*-to-*1. *That is,* $\mathrm{card}\{x \in X : T(x) = y\} \le N$ *for every* $y \in Y$. *Then for every set of real numbers* $\{a_y\}_{y \in Y}$ *we have*

$$\Phi_p^-(\{a_{T(x)}\}_{x \in X}) \le \max\{p, 2\} N^{1/p} \Phi_p^-(\{a_y\}_{y \in Y}).$$

We will now bring the radial contractions ρ_ℓ defined by (4.1) into our estimates. Recall that the index set I was defined in Section 3 and that for each $m \in \mathbf{Z}_+$, we write

$$I_m = \{(k, j) \in I : k \le m\}.$$

Lemma 4.3. *There exists a constant* $C_{4.3}$ *which depends only on the complex dimension* n *such that the following holds true: Let* $h : \mathbf{B} \to [0, \infty)$ *be a map such that* $\sup_{w \in T_{k,j}} h(w) < \infty$ *for every* $(k, j) \in I$. *For each* $(k, j) \in I$, *let* $w_{k,j} \in T_{k,j}$ *be such that*

$$h(w_{k,j}) \ge \frac{1}{2} \sup_{w \in T_{k,j}} h(w). \tag{4.3}$$

Suppose that $z_{k,j} \in T_{k,j}$ *for every* $(k, j) \in I$. *Then the inequality*

$$\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j) \in I_m}) \le \max\{p, 2\} C_{4.3} 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j) \in I_m})$$

holds for all $m, \ell \in \mathbf{N}$ *and* $1 < p < \infty$.

Proof. First of all, by (3.3) and (3.2), there exists a natural number C_1 such that for all integers $0 \le k' \le k$ and $1 \le i \le m(k')$, we have

$$\mathrm{card}\{j \in \{1, \ldots, m(k)\} : B(u_{k,j}, 2^{-k}) \cap B(u_{k',i}, 2^{-k'}) \ne \emptyset\} \le C_1 2^{2n(k-k')}. \tag{4.4}$$

Let h, $w_{k,j}$ and $z_{k,j}$, $(k, j) \in I$, be as in the statement of the lemma. Let $\ell \in \mathbf{N}$. By (4.2) and (3.5), we have

$$\rho_\ell \left(\cup_{j=1}^{m(k)} T_{k,j} \right) \subset \cup_{i=1}^{m(k-\ell)} T_{k-\ell,i} \quad \text{if} \quad k > \ell. \tag{4.5}$$

Consider any $1 < p < \infty$ and $m \in \mathbf{N}$. First let us consider the case where $m > \ell$. Then $I_m = I_\ell \cup I_{m,\ell}$, where

$$I_{m,\ell} = \{(k, j) \in I : \ell < k \le m\}.$$

By (4.5), for each $(k, j) \in I_{m,\ell}$, there is an $\eta(k, j) \in \{1, \ldots, m(k - \ell)\}$ such that $\rho_\ell(z_{k,j}) \in T_{k-\ell,\eta(k,j)}$. We now define a map $\varphi : I_{m,\ell} \to I_m$ by the formula

$$\varphi(k, j) = (k - \ell, \eta(k, j)), \quad (k, j) \in I_{m,\ell}.$$

This map ensures that $\rho_\ell(z_{k,j}) \in T_{\varphi(k,j)}$, $(k,j) \in I_{m,\ell}$. By (4.3), we have

$$h(\rho_\ell(z_{k,j})) \le 2h(w_{\varphi(k,j)}) \quad \text{for every} \quad (k,j) \in I_{m,\ell}.$$

Consequently,

$$\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_{m,\ell}}) \le 2\Phi_p^-(\{h(w_{\varphi(k,j)})\}_{(k,j)\in I_{m,\ell}}). \tag{4.6}$$

By (4.1), if $(k,j) \in I_{m,\ell}$, then

$$\frac{\rho_\ell(z_{k,j})}{|\rho_\ell(z_{k,j})|} = \frac{z_{k,j}}{|z_{k,j}|}.$$

Since $z_{k,j} \in T_{k,j}$ and $\rho_\ell(z_{k,j}) \in T_{\varphi(k,j)} = T_{k-\ell,\eta(k,j)}$, by (3.5), the above identity implies

$$B(u_{k,j}, 2^{-k}) \cap B(u_{k-\ell,\eta(k,j)}, 2^{-k+\ell}) \ne \emptyset.$$

Combining this with (4.4), we see that for each $i \in \{1, \ldots, m(k-\ell)\}$,

$$\operatorname{card}\{j \in \{1, \ldots, m(k)\} : \eta(k,j) = i\} \le C_1 2^{2n\ell}.$$

In other words, the map $\varphi : I_{m,\ell} \to I_m$ is at most $C_1 2^{2n\ell}$-to-1. By Lemma 4.2, this means

$$\Phi_p^-(\{h(w_{\varphi(k,j)})\}_{(k,j)\in I_{m,\ell}}) \le \max\{p,2\} C_1^{1/p} 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_m}).$$

Since $C_1^{1/p} \le C_1$, if we combine the above with (4.6), we obtain

$$\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_{m,\ell}}) \le \max\{p,2\} 2C_1 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_m}). \tag{4.7}$$

Next we consider the set I_ℓ.

Note that by (3.3) and (3.2), there is a natural number C_2 such that

$$m(k) \le C_2 2^{2nk} \quad \text{for every} \quad k \ge 0. \tag{4.8}$$

By (4.1), we have

$$\rho_\ell\left(\cup_{j=1}^{m(k)} T_{k,j}\right) \subset \cup_{i=1}^{m(0)} T_{0,i} \quad \text{if} \quad 0 \le k \le \ell.$$

Therefore there is a map $\psi : I_\ell \to I_0$ such that

$$\rho_\ell(z_{k,j}) \in T_{\psi(k,j)} \quad \text{for every} \quad (k,j) \in I_\ell.$$

Combining this relation with (4.3), we have

$$\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_\ell}) \le 2\Phi_p^-(\{h(w_{\psi(k,j)})\}_{(k,j)\in I_\ell}).$$

By (4.8), $\mathrm{card}(I_\ell) \le C_2 \sum_{k=0}^{\ell} 2^{2nk} \le 2C_2 2^{2n\ell}$. Therefore the map $\psi : I_\ell \to I_0$ is at most $2C_2 2^{2n\ell}$-to-1. Applying Lemma 4.2 again, we obtain

$$\Phi_p^-(\{h(w_{\psi(k,j)})\}_{(k,j)\in I_\ell}) \le \max\{p,2\} 2C_2 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_0}).$$

Therefore

$$\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_\ell}) \le \max\{p,2\} 4C_2 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_0}). \tag{4.9}$$

Combining this with (4.7), we see that in the case $m > \ell$ we have

$$\begin{aligned}\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_m}) &\le \Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_{m,\ell}}) + \Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_\ell}) \\ &\le \max\{p,2\}(2C_1 + 4C_2) 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_m}).\end{aligned}$$

On the other hand, if $m \le \ell$, then (4.9) gives us

$$\begin{aligned}\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_m}) &\le \Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_\ell}) \\ &\le \max\{p,2\} 4C_2 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_0}) \\ &\le \max\{p,2\} 4C_2 2^{2n\ell/p} \Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_m}).\end{aligned}$$

This completes the proof of the lemma. □

Proposition 4.4. *Given any $2n < p < \infty$, there exists a constant $C_{4.4}(p)$ which depends only on p and the complex dimension n such that the following estimate holds: Let $f \in L^2(S, d\sigma)$. For each $(k,j) \in I$, let $w_{k,j} \in T_{k,j}$ be such that*

$$\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\| \ge \frac{1}{2} \sup_{w \in T_{k,j}} \|M_{m_w} H_f k_w\|. \tag{4.10}$$

Let $z_{k,j} \in T_{k,j}$, $(k,j) \in I$. Then for every $m \in \mathbf{N}$ we have

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) \le C_{4.4}(p) \Phi_p^-(\{\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\|\}_{(k,j)\in I_m}). \tag{4.11}$$

Proof. Since $p > 2n$, there is a $0 < \delta \le 1/2$ such that if we set

$$\epsilon = 1 - \delta - (2n/p),$$

then $\epsilon > 0$. Let $f \in L^2(S, d\sigma)$, and let $w_{k,j}$ and $z_{k,j}$ be as in the statement of the proposition. By Theorem 4.1, we have

$$\mathrm{Var}^{1/2}(f - Pf; z_{k,j}) \le C(\delta) \sum_{\ell=1}^{\infty} \frac{1}{2^{(1-\delta)\ell}} \|M_{m_{\rho_\ell(z_{k,j})}} H_f k_{\rho_\ell(z_{k,j})}\|$$

for every $(k,j) \in I$. Since Φ_p^- is a norm on $\hat{c}$, it follows that

$$
\Phi_p^-(\{\mathrm{Var}^{1/2}(f-Pf;z_{k,j})\}_{(k,j)\in I_m}) \le
$$

$$
C(\delta)\sum_{\ell=1}^{\infty}\frac{1}{2^{(1-\delta)\ell}}\Phi_p^-(\{\|M_{m_{\rho_\ell(z_{k,j})}}H_f k_{\rho_\ell(z_{k,j})}\|\}_{(k,j)\in I_m}) \tag{4.12}
$$

for every $m \in \mathbf{N}$. Next, we define

$$
h(w) = \|M_{m_w}H_f k_w\|, \quad w \in \mathbf{B}.
$$

Then (4.10) tells us that this map $h : \mathbf{B} \to [0,\infty)$ and the points $w_{k,j}$, $(k,j) \in I$, satisfy condition (4.3). This allows us to apply Lemma 4.3 to obtain

$$
\begin{aligned}
\sum_{\ell=1}^{\infty}\frac{1}{2^{(1-\delta)\ell}}\Phi_p^-(\{\|M_{m_{\rho_\ell(z_{k,j})}}H_f k_{\rho_\ell(z_{k,j})}\|\}_{(k,j)\in I_m}) &= \sum_{\ell=1}^{\infty}\frac{1}{2^{(1-\delta)\ell}}\Phi_p^-(\{h(\rho_\ell(z_{k,j}))\}_{(k,j)\in I_m}) \\
&\le pC_{4.3}\sum_{\ell=1}^{\infty}\frac{2^{2n\ell/p}}{2^{(1-\delta)\ell}}\Phi_p^-(\{h(w_{k,j})\}_{(k,j)\in I_m}) \\
&= pC_{4.3}\sum_{\ell=1}^{\infty}\frac{1}{2^{\epsilon\ell}}\Phi_p^-(\{\|M_{m_{w_{k,j}}}H_f k_{w_{k,j}}\|\}_{(k,j)\in I_m}).
\end{aligned}
$$

Combining this with (4.12), we see that the proposition holds for the constant $C_{4.4} = pC(\delta)C_{4.3}\sum_{\ell=1}^{\infty}2^{-\epsilon\ell}$. □

5. Lower bound

Propositions 3.7 and 4.4 represent the two main steps in the proof of the lower bound in Theorem 1.2. The remaining step in the proof of the lower bound is to bridge the gap between the right-hand side of (4.11) and the left-hand side of (3.17), which only involves existing ideas. Nonetheless, we repeat all the necessary details here for completeness.

Lemma 5.1. [6,Lemma 5.2] *There is a constant* $0 < C_{5.1} < \infty$ *such that*

$$
\|M_{m_z}H_f k_z\| \le \|H_f\psi_{z,t}\| + C_{5.1}t\mathrm{Var}^{1/2}(f-Pf;z),
$$

for all $f \in L^2(S,d\sigma)$, $z \in \mathbf{B}$ *and* $0 < t \le 1$.

Proposition 5.2. *Given any* $2n < p < \infty$, *there is a constant* $C_{5.2}(p)$ *such that the following holds true: Let* $f \in L^2(S,d\sigma)$. *For each* (k,j), *let* $z_{k,j} \in T_{k,j}$ *satisfy the condition*

$$
\mathrm{Var}(f-Pf;z_{k,j}) \ge \frac{1}{2}\sup_{z\in T_{k,j}}\mathrm{Var}(f-Pf;z). \tag{5.1}
$$

Then

$$
\Phi_p^-(\{\mathrm{Var}^{1/2}(f-Pf;z_{k,j})\}_{(k,j)\in I}) \le C_{5.2}(p)\|H_f\|_p^-.
$$

Proof. Let $f \in L^2(S, d\sigma)$ be given. For each $(k,j) \in I$, we pick a $w_{k,j} \in T_{k,j}$ such that

$$\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\| \geq \frac{1}{2} \sup_{w \in T_{k,j}} \|M_{m_w} H_f k_w\|.$$

Then by Proposition 4.4 we have

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) \leq C_{4.4}(p)\Phi_p^-(\{\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\|\}_{(k,j)\in I_m})$$

for every $m \in \mathbf{N}$. Applying Lemma 5.1 to each $\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\|$, for $0 < t \leq 1$ we have

$$\begin{aligned}\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) &\leq C_{4.4}(p)\Phi_p^-(\{\|H_f \psi_{w_{k,j},t}\|\}_{(k,j)\in I_m}) \\ &\quad + C_{4.4}(p)C_{5.1}t\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; w_{k,j})\}_{(k,j)\in I_m}).\end{aligned}$$

Since $w_{k,j} \in T_{k,j}$, it follows from (5.1) that $\mathrm{Var}(f - Pf; w_{k,j}) \leq 2\mathrm{Var}(f - Pf; z_{k,j})$. Hence

$$\begin{aligned}\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) &\leq C_{4.4}(p)\Phi_p^-(\{\|H_f \psi_{w_{k,j},t}\|\}_{(k,j)\in I_m}) \\ &\quad + 2C_{4.4}(p)C_{5.1}t\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}).\end{aligned}$$

Now, for the given $2n < p < \infty$, we pick $0 < t \leq 1$ such that $2C_{4.4}(p)C_{5.1}t \leq 1/2$. This fixes the value of t in terms of p, and from the above inequality we obtain

$$\begin{aligned}\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) &\leq C_{4.4}(p)\Phi_p^-(\{\|H_f \psi_{w_{k,j},t}\|\}_{(k,j)\in I_m}) \\ &\quad + (1/2)\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}).\end{aligned}$$

Since I_m is a finite set, the quantity $\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m})$ is finite. Therefore after the obvious cancellation the above inequality becomes

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) \leq 2C_{4.4}(p)\Phi_p^-(\{\|H_f \psi_{w_{k,j},t}\|\}_{(k,j)\in I_m}).$$

Assuming $\|H_f\|_p^- < \infty$, an application of Proposition 3.7 to the right-hand side gives us

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I_m}) \leq 2C_{4.4}(p)C_{3.4}(t,p)M_{3.6}\|H_f\|_p^-.$$

Since this holds for every $m \in \mathbf{N}$, by (1.1) we have

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I}) \leq 2C_{4.4}(p)C_{3.4}(t,p)M_{3.6}\|H_f\|_p^-.$$

Thus the proposition holds for the constant $C_{5.2}(p) = 2C_{4.4}(p)C_{3.4}(t,p)M_{3.6}$. $\square$

Lemma 5.3. [18,Lemma 2.4] *Given any* $0 < a < \infty$, *there exists a natural number* K *which depends only on* a *and the complex dimension* n *such that the following holds true*: *Suppose that* Γ *is an* a*-separated subset of* $\mathbf{B}$. *Then there exist pairwise disjoint subsets* $\Gamma_1, \dots, \Gamma_K$ *of* Γ *such that* $\cup_{i=1}^K \Gamma_i = \Gamma$ *and such that* $\mathrm{card}(\Gamma_i \cap T_{k,j}) \leq 1$ *for all* $i \in \{1, \dots, K\}$ *and* $(k,j) \in I$.

With the above preparation, we now have

Proof of the lower bound in Theorem 1.2. Let $2n < p < \infty$ and $a > 0$ be given. We need to find a $0 < C_1 < \infty$ that depends only on p, a and n such that the inequality

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma}) \le C_1\|H_f\|_p^- \tag{5.2}$$

holds for every $f \in L^2(S, d\sigma)$ and every a-separated set Γ in $\mathbf{B}$.

Let an a-separated set Γ in $\mathbf{B}$ be given. Then Lemma 5.3 provides the partition

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_K \tag{5.3}$$

where K depends only on a and n, such that

$$\mathrm{card}(\Gamma_i \cap T_{k,j}) \le 1 \quad \text{for all} \quad i \in \{1, \dots, K\} \quad \text{and} \quad (k, j) \in I. \tag{5.4}$$

Let $f \in L^2(S, d\sigma)$. For each $(k, j) \in I$ pick $z_{k,j} \in T_{k,j}$ such that (5.1) holds. Combining (5.1) with (5.4), we see that

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma_i}) \le 2\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I})$$

for every $i \in \{1, \dots, K\}$. Proposition 5.2 tells us that

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I}) \le C_{5.2}(p)\|H_f\|_p^-.$$

Therefore

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma_i}) \le 2C_{5.2}(p)\|H_f\|_p^-,$$

$i \in \{1, \dots, K\}$. By (5.3) and (3.18) we have

$$\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma}) \le \sum_{i=1}^{K} \Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma_i}).$$

By the above two inequalities, (5.2) holds for the constant $C_1 = 2KC_{5.2}(p)$. $\square$

6. Small factor and cancellation

We now turn to the upper bound in Theorem 1.2. One of the main ingredients in the proof of the upper bound is a reverse Hölder's inequality, an inequality that is analogous to Proposition 6.4 in [6]. But whereas [6,Proposition 6.4] works for the symmetric gauge function Φ_p^+, here the inequality must cover Φ_p^-, which makes its proof a much more difficult task. The reader will see that the key to the proof of the reverse Hölder's inequality is a certain cancellation, and what enables this cancellation to take place is a certain "small factor". Here we must take an approach that is fundamentally different from the corresponding part in [6] to obtain the requisite "small factor".

For any $a = \{a_j\}_{j\in\mathbf{N}}$ and $N \in \mathbf{N}$, define the sequence $a^{[N]} = \{a_j^N\}_{j\in\mathbf{N}}$ by the formula

$$a_j^N = a_i \quad \text{if} \quad (i-1)N + 1 \le j \le iN,\ i \in \mathbf{N}. \tag{6.1}$$

In other words, $a^{[N]}$ is obtained from a by repeating each term N times. Alternately, we can think of $a^{[N]}$ as $a \oplus \cdots \oplus a$, the "direct sum" of N copies of a.

Definition 6.1. [16,Definition 2.2] A symmetric gauge function Φ is said to satisfy condition (DQK) if there exist constants $0 < \theta < 1$ and $0 < \alpha < \infty$ such that

$$\Phi(a^{[N]}) \ge \alpha N^\theta \Phi(a)$$

for every $a \in \hat{c}$ and every $N \in \mathbf{N}$.

The relevance of Definition 6.1 to what we do in this paper is the following:

Lemma 6.2. *For each $1 < p < \infty$, the symmetric gauge function Φ_p^- satisfies condition* (DQK). *More precisely, we have $\Phi_p^-(a^{[N]}) \ge N^{1/p}\Phi_p^-(a)$ for all $a \in \hat{c}$ and $N \in \mathbf{N}$.*

Proof. Let $1 < p < \infty$. It suffices to consider $a = \{a_j\}_{j\in\mathbf{N}}$ where the terms are non-negative and in descending order, i.e.,

$$a_1 \ge a_2 \ge \cdots \ge a_j \ge \cdots .$$

Then by (6.1) and the definition of Φ_p^-, for every $N \in \mathbf{N}$ we have

$$\Phi_p^-(a^{[N]}) = \sum_{i=1}^\infty a_i \sum_{j=1}^N \frac{1}{((i-1)N + j)^{(p-1)/p}} \ge \sum_{i=1}^\infty \frac{a_i N}{(iN)^{(p-1)/p}} = N^{1/p}\Phi_p^-(a)$$

as promised. □

The proof of the reverse Hölder's inequality for Φ_p^- will be based on condition (DQK). But for the proof itself it will be more convenient to work with (DQK), rather than with the specific Φ_p^-.

Recall that for each $k \ge 0$, in Section 3 we introduced $\{u_{k,1}, \dots, u_{k,m(k)}\}$, which is a subset of S that is maximal with respect to (3.3). For each $(k, j) \in I$, we now define

$$A_{k,j} = B(u_{k,j}, 2^{-k+1}), \quad B_{k,j} = B(u_{k,j}, 2^{-k+2}) \quad \text{and} \quad C_{k,j} = B(u_{k,j}, 2^{-k+3}). \tag{6.2}$$

Definition 6.3. For each $i \in \mathbf{Z}_+$ and each $(k, j) \in I$, we set

$$E_i(k, j) = \{(k + i, j') \in I : A_{k+i,j'} \cap B_{k,j} \ne \emptyset\}.$$

Definition 6.4. Suppose that $g \in L^2(S, d\sigma)$.

(a) For each $1 \le t < \infty$ and each $(k,j) \in I$, define

$$J_t(g;k,j) = \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |g - g_{B_{k,j}}|^t d\sigma \right)^{1/t}.$$

(b) For each $k \in \mathbf{Z}_+$, define the function $R_k g$ on S by the formula

$$(R_k g)(\zeta) = \frac{1}{\sigma(B(\zeta, 2^{-k-2}))} \int_{B(\zeta,2^{-k-2})} g d\sigma, \quad \zeta \in S.$$

(c) For $1 \le t < \infty$, $i \in \mathbf{Z}_+$ and $(k,j) \in I$, define

$$G_{t,i}(g;k,j) = \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |g - R_{k+i}g|^t d\sigma \right)^{1/t} \quad \text{and}$$

$$H_{t,i}(g;k,j) = \left(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |R_{k+i}g - g_{B_{k,j}}|^t d\sigma \right)^{1/t}.$$

(d) For each $(k,j) \in I$, define

$$J(g;k,j) = \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma.$$

Lemma 6.5. *There is a constant $C_{6.5}$ such that*

$$G^t_{t,i}(g;k,j) \le 2^{t-1} C_{6.5} \sum_{(k+i,j') \in E_i(k,j)} \frac{\sigma(B_{k+i,j'})}{\sigma(B_{k,j})} J^t_t(g;k+i,j')$$

for all $g \in L^2(S, d\sigma)$, $1 \le t < \infty$, $i \in \mathbf{Z}_+$ and $(k,j) \in I$.

Proof. By (3.2) and (6.2), there is a constant C_1 such that

$$\frac{\sigma(B_{k+i,j'})}{\sigma(B(\zeta, 2^{-k-i-2}))} \le C_1$$

for all $k, i \in \mathbf{Z}_+$, $j' \in \{1, \dots, m(k+i)\}$ and $\zeta \in S$. Let $g \in L^2(S, d\sigma)$, $1 \le t < \infty$, $i \in \mathbf{Z}_+$ and $(k,j) \in I$. Then by Definition 6.3 and (3.4) we have

$$\begin{aligned}
\int_{B_{k,j}} |g - R_{k+i}g|^t d\sigma &\le \sum_{(k+i,j') \in E_i(k,j)} \int_{A_{k+i,j'}} |g - R_{k+i}g|^t d\sigma \\
&\le 2^{t-1} \sum_{(k+i,j') \in E_i(k,j)} \int_{B_{k+i,j'}} |g - g_{B_{k+i,j'}}|^t d\sigma \\
&+ 2^{t-1} \sum_{(k+i,j') \in E_i(k,j)} \int_{A_{k+i,j'}} |g_{B_{k+i,j'}} - R_{k+i}g|^t d\sigma. \qquad (6.3)
\end{aligned}$$

For each $\zeta \in A_{k+i,j'}$ we have $B(\zeta, 2^{-k-i-2}) \subset B_{k+i,j'}$. Therefore

$$
\begin{aligned}
|g_{B_{k+i,j'}} - R_{k+i}g(\zeta)|^t &\le \frac{1}{\sigma(B(\zeta, 2^{-k-i-2}))} \int_{B(\zeta,2^{-k-i-2})} |g_{B_{k+i,j'}} - g|^t d\sigma \\
&\le C_1 J_t^t(g; k+i, j')
\end{aligned}
$$

for every $\zeta \in A_{k+i,j'}$. Hence

$$
\int_{A_{k+i,j'}} |g_{B_{k+i,j'}} - R_{k+i}g|^t d\sigma \le C_1 \sigma(A_{k+i,j'}) J_t^t(g; k+i, j').
$$

Substituting this in (6.3), we see that if we set $C_{6.5} = 1 + C_1$, then the lemma holds. $\square$

Lemma 6.6. [18,Lemma 2.2] *Suppose that X and Y are countable sets and that N is a natural number. Suppose that $T : X \to Y$ is a map that is at most N-to-1. That is, for every $y \in Y$, $\operatorname{card}\{x \in X : T(x) = y\} \le N$. Then for every set of real numbers $\{b_y\}_{y \in Y}$ and every symmetric gauge function Φ, we have*

$$
\Phi(\{b_{T(x)}\}_{x \in X}) \le N\Phi(\{b_y\}_{y \in Y}).
$$

The next lemma is the most crucial step in the proof of our reverse Hölder's inequality: extraction of the requisite "small factor".

Lemma 6.7. *Let Φ be a symmetric gauge function satisfying condition* (DQK). *Let $1 \le t < \infty$ and $\epsilon > 0$ also be given. Then there exists a natural number $\nu \in \mathbf{N}$ which depends only on Φ, t, ϵ and the complex dimension n such that*

$$
\Phi(\{G_{t,\nu}(g; k, j)\}_{(k,j) \in I_m}) \le \epsilon \Phi(\{J_t(g; k, j)\}_{(k,j) \in I_{m+\nu}})
$$

for all $g \in L^2(S, d\sigma)$ and $m \in \mathbf{N}$.

Proof. We begin by fixing a number of necessary constants. First of all, by (3.3) and (3.2), there is a natural number $M_1 \in \mathbf{N}$ such that

$$
\operatorname{card}\{j' \in \{1, \ldots, m(k)\} : C_{k,j'} \cap C_{k,j} \ne \emptyset\} \le M_1 \tag{6.4}
$$

for every $(k, j) \in I$. Let $m \in \mathbf{N}$. By a standard maximality argument, there is a partition

$$
I_m = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{M_1} \tag{6.5}
$$

of the truncated index set I_m such that for each pair of $q \in \{1, \ldots, M_1\}$ and $k \in \mathbf{Z}_+$,

$$
\text{if } \ (k, j), (k, j') \in \mathcal{I}_q \ \text{ and } \ j \ne j', \ \text{ then } \ C_{k,j} \cap C_{k,j'} = \emptyset. \tag{6.6}
$$

Again by (3.3) and (3.2), there are constants $0 < c_1 \le C_1 < \infty$ such that

$$
c_1 2^{-2ni} \le \frac{\sigma(B(\zeta, 2^{-i}r))}{\sigma(B(\xi, r))} \le C_1 2^{-2ni} \tag{6.7}
$$

holds for all $\zeta, \xi \in S$, $0 < r \le 8$ and $i \in \mathbf{Z}_+$. In particular, we have $\sigma(B_{k,j}) \le C_2\sigma(A_{k,j})$ and $\sigma(C_{k,j}) \le C_2\sigma(B_{k,j})$ for every $(k,j) \in I$, where $C_2 = (2^{2n}/c_1)$. Note that for every $i \in \mathbf{Z}_+$, if $(k+i, j') \in E_i(k,j)$, then $A_{k+i,j'} \subset C_{k,j}$. Combining these facts with (6.4), we see that if we set $C_3 = M_1 C_2^2$, then

$$\sum_{(k+i,j')\in E_i(k,j)} \frac{\sigma(B_{k+i,j'})}{\sigma(B_{k,j})} \le C_3 \tag{6.8}$$

for all $i \in \mathbf{Z}_+$ and $(k,j) \in I$.

Suppose that Φ is a symmetric gauge function satisfying condition (DQK). Then Definition 6.1 implies that there exist constants $0 < \theta < 1$ and $0 < C_4 < \infty$ such that

$$\Phi(a) \le C_4 N^{-\theta}\Phi(a^{[N]}) \quad \text{for all} \;\; a \in \hat{c} \;\; \text{and} \;\; N \in \mathbf{N}. \tag{6.9}$$

Let $1 \le t < \infty$ be given. We write $C_5 = 2^{t-1}C_{6.5}$, where $C_{6.5}$ is the constant provided by Lemma 6.5. Let $\epsilon > 0$ also be given. We pick an $N_0 \in \mathbf{N}$ such that

$$(4C_3)^{1/t} C_4 N_0^{-\theta} \le \frac{\epsilon}{2M_1 C_5^{1/t}}. \tag{6.10}$$

Finally, with N_0 so chosen, we pick a $\nu \in \mathbf{N}$ such that

$$(4N_0 C_1 C_5 2^{-2n\nu})^{1/t} M_1 \le \epsilon/2. \tag{6.11}$$

What remains is to show that the lemma holds for this ν.

Let $g \in L^2(S, d\sigma)$ be given. It suffice to consider the case where $J_t(g;k,j) < \infty$ for every $(k,j) \in I_{m+\nu}$. For each $(k,j) \in I_m$, Lemma 6.5 gives us

$$\begin{aligned} G_{t,\nu}^t(g;k,j) &\le C_5 \sum_{(k+\nu,j')\in E_\nu(k,j)} \frac{\sigma(B_{k+\nu,j'})}{\sigma(B_{k,j})} J_t^t(g;k+\nu,j') \\ &= C_5 \sum_{(k+\nu,j')\in \tilde{E}_\nu(k,j)} \frac{\sigma(B_{k+\nu,j'})}{\sigma(B_{k,j})} J_t^t(g;k+\nu,j'), \end{aligned} \tag{6.12}$$

where $\tilde{E}_\nu(k,j) = \{(k+\nu, j') \in E_\nu(k,j) : J_t^t(g;k+\nu,j') > 0\}$. Now, for every $(k,j) \in I_m$, we have the decomposition

$$\tilde{E}_\nu(k,j) = \bigcup_{\ell=-\infty}^{\infty} X_\ell(k,j),$$

where $X_\ell(k,j)$ is the collection of $(k+\nu, j') \in E_\nu(k,j)$ satisfying the condition

$$2^{\ell-1} < J_t^t(g;k+\nu,j') \le 2^\ell, \tag{6.13}$$

$\ell \in \mathbf{Z}$. For each $(k, j) \in I_m$, define the sets

$$Z^{(1)}(k, j) = \{\ell \in \mathbf{Z} : 1 \leq \operatorname{card}(X_\ell(k, j)) \leq N_0\} \quad \text{and}$$
$$Z^{(2)}(k, j) = \{\ell \in \mathbf{Z} : \operatorname{card}(X_\ell(k, j)) > N_0\}.$$

It follows from (6.12) that

$$G^t_{t,\nu}(g; k, j) \leq C_5\{T^{(1)}(k, j) + T^{(2)}(k, j)\}, \tag{6.14}$$

where, for $i = 1, 2$,

$$T^{(i)}(k, j) = \sum_{\ell \in Z^{(i)}(k,j)} \sum_{(k+\nu,j') \in X_\ell(k,j)} \frac{\sigma(B_{k+\nu,j'})}{\sigma(B_{k,j})} J^t_t(g; k + \nu, j').$$

Let us first consider $T^{(1)}(k, j)$. Suppose that $(k, j) \in I_m$ is such that $Z^{(1)}(k, j) \neq \emptyset$. Since $E_\nu(k, j)$ is a finite set, the set $Z^{(1)}(k, j)$ is also finite and, consequently, has a largest element $\mu(k, j)$. Thus there is an $\eta(k, j) \in \{1, \ldots, m(k+\nu)\}$ such that $(k+\nu, \eta(k, j)) \in X_{\mu(k,j)}(k, j)$. By (6.13), we have

$$2^{\mu(k,j)} \leq 2J^t_t(g; k + \nu, \eta(k, j)).$$

By (6.7), $\sigma(B_{k+\nu,j'})/\sigma(B_{k,j}) \leq C_1 2^{-2n\nu}$. Since $\operatorname{card}(X_\ell(k, j)) \leq N_0$ for every $\ell \in Z^{(1)}(k, j)$ and since $\mu(k, j)$ is the largest element in $Z^{(1)}(k, j)$, we have

$$\begin{aligned} T^{(1)}(k, j) &\leq \sum_{\ell=-\infty}^{\mu(k,j)} N_0 C_1 2^{-2n\nu} 2^\ell = 2N_0 C_1 2^{-2n\nu} 2^{\mu(k,j)} \\ &\leq 4N_0 C_1 2^{-2n\nu} J^t_t(g; k + \nu, \eta(k, j)). \end{aligned}$$

If $(k, j) \in I_m$ is such that $Z^{(1)}(k, j) = \emptyset$, then $T^{(1)}(k, j) = 0$. Thus we conclude that for every $(k, j) \in I_m$, there is an $\eta(k, j) \in \{1, \ldots, m(k+\nu)\}$ such that $(k+\nu, \eta(k, j)) \in E_\nu(k, j)$ and such that

$$T^{(1)}(k, j) \leq 4N_0 C_1 2^{-2n\nu} J^t_t(g; k + \nu, \eta(k, j)). \tag{6.15}$$

Now define the map $\varphi : I_m \to I_{m+\nu}$ by the formula

$$\varphi(k, j) = (k + \nu, \eta(k, j)),$$

$(k, j) \in I_m$. If $k \in \mathbf{Z}_+$ and $j_1, j_2 \in \{1, \ldots, m(k)\}$ are such that $\eta(k, j_1) = \eta(k, j_2)$, then, by the definition of η we have $E_\nu(k, j_1) \cap E_\nu(k, j_2) \neq \emptyset$. By (6.2), if $A_{k+i,j'} \cap B_{k,j} \neq \emptyset$, then $A_{k+i,j'} \subset C_{k,j}$. Hence the condition $E_\nu(k, j_1) \cap E_\nu(k, j_2) \neq \emptyset$ implies $C_{k,j_1} \cap C_{k,j_2} \neq \emptyset$. By (6.4), the map $\varphi : I_m \to I_{m+\nu}$ is at most M_1-to-1. Thus Lemma 6.6 gives us

$$\begin{aligned} \Phi(\{J_t(g; k + \nu, \eta(k, j))\}_{(k,j) \in I_m}) &= \Phi(\{J_t(g; \varphi(k, j))\}_{(k,j) \in I_m}) \\ &\leq M_1 \Phi(\{J_t(g; k, j)\}_{(k,j) \in I_{m+\nu}}). \end{aligned}$$

By (6.15), we have

$$(T^{(1)}(k,j))^{1/t} \le (4N_0C_1 2^{-2n\nu})^{1/t} J_t(g;k+\nu,\eta(k,j))$$

for every $(k,j) \in I_m$. The combination of these two inequalities gives us

$$\Phi(\{(T^{(1)}(k,j))^{1/t}\}_{(k,j)\in I_m}) \le (4N_0C_1 2^{-2n\nu})^{1/t} M_1 \Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}). \tag{6.16}$$

It follows from (6.14) that

$$G_{t,\nu}(g;k,j) \le C_5^{1/t}\{(T^{(1)}(k,j))^{1/t} + (T^{(2)}(k,j))^{1/t}\}.$$

Hence

$$\begin{aligned}
&\Phi(\{G_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) \\
&\qquad \le C_5^{1/t}\Phi(\{(T^{(1)}(k,j))^{1/t}\}_{(k,j)\in I_m}) + C_5^{1/t}\Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in I_m}) \\
&\qquad \le (\epsilon/2)\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) + C_5^{1/t}\Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in I_m}),
\end{aligned}$$

where the second $\le$ follows from (6.16) and (6.11). Thus the proof of the lemma is reduced to the proof of the inequality

$$C_5^{1/t}\Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in I_m}) \le (\epsilon/2)\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}).$$

By (6.5) and (3.18), this inequality will follow if we can show that

$$\Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in \mathcal{I}_q}) \le \frac{\epsilon}{2M_1C_5^{1/t}}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) \tag{6.17}$$

for every $q \in \{1,\dots,M_1\}$.

To prove (6.17), consider any $q \in \{1,\dots,M_1\}$, and define $\tilde{\mathcal{I}}_q = \{(k,j) \in \mathcal{I}_q : Z^{(2)}(k,j) \ne \emptyset\}$. Again, each $Z^{(2)}(k,j)$ is a finite set because $\mathrm{card}(E_\nu(k,j)) < \infty$. Thus for each $(k,j) \in \tilde{\mathcal{I}}_q$, $Z^{(2)}(k,j)$ has a largest element $\lambda(k,j)$. That is,

$$\mathrm{card}(X_{\lambda(k,j)}(k,j)) > N_0, \tag{6.18}$$

and $\ell \notin Z^{(2)}(k,j)$ if $\ell > \lambda(k,j)$. For each $(k,j) \in \tilde{\mathcal{I}}_q$, pick an $h(k,j) \in \{1,\dots,m(k+\nu)\}$ such that

$$(k+\nu, h(k,j)) \in X_{\lambda(k,j)}(k,j).$$

Since $\lambda(k,j)$ is the largest element in $Z^{(2)}(k,j)$, by (6.13) we have

$$J_t^t(g;k+\nu,j') \le 2J_t^t(g;k+\nu,h(k,j)) \quad \text{for every} \quad (k+\nu,j') \in \bigcup_{\ell\in Z^{(2)}(k,j)} X_\ell(k,j).$$

Combining this with the definition of $T^{(2)}(k,j)$ and with (6.8), we obtain

$$T^{(2)}(k,j) \le 2C_3 J_t^t(g; k+\nu, h(k,j)).$$

Thus $(T^{(2)}(k,j))^{1/t} \le (2C_3)^{1/t} J_t(g; k+\nu, h(k,j))$ for every $(k,j) \in \tilde{\mathcal{I}}_q$. Consequently,

$$\begin{aligned} \Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in\mathcal{I}_q}) &= \Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in\tilde{\mathcal{I}}_q}) \\ &\le (2C_3)^{1/t}\Phi(\{J_t(g; k+\nu, h(k,j))\}_{(k,j)\in\tilde{\mathcal{I}}_q}). \end{aligned} \tag{6.19}$$

Recall that the condition $A_{k+i,j'} \cap B_{k,j} \ne \emptyset$ implies $A_{k+i,j'} \subset C_{k,j}$. Combining this fact with (6.6), we have $E_\nu(k_1,j_1) \cap E_\nu(k_2,j_2) = \emptyset$ for all $(k_1,j_1) \ne (k_2,j_2)$ in $\tilde{\mathcal{I}}_q$. Therefore

$$X_{\lambda(k_1,j_1)}(k_1,j_1) \cap X_{\lambda(k_2,j_2)}(k_2,j_2) = \emptyset \quad \text{for all} \quad (k_1,j_1) \ne (k_2,j_2) \quad \text{in} \quad \tilde{\mathcal{I}}_q. \tag{6.20}$$

Note that (6.13) also gives us

$$J_t(g; k+\nu, h(k,j)) \le 2^{1/t} J_t(g; k+\nu, j') \quad \text{for every} \quad (k+\nu, j') \in X_{\lambda(k,j)}(k,j). \tag{6.21}$$

If $(k,j) \in \tilde{\mathcal{I}}_q$, then, of course, $X_{\lambda(k,j)}(k,j) \subset I_{m+\nu}$. Thus, if we re-enumerate the numbers $\{J_t(g; k+\nu, h(k,j))\}_{(k,j)\in\tilde{\mathcal{I}}_q}$ in the form $b = \{b_1, \dots, b_i\}$, then it follows from the combination of (6.21), (6.20) and (6.18) that

$$\Phi(b^{[N_0]}) \le 2^{1/t}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}).$$

Applying (6.9), we now have

$$\begin{aligned} \Phi(\{J_t(g; k+\nu, h(k,j))\}_{(k,j)\in\tilde{\mathcal{I}}_q}) &= \Phi(b) \le C_4 N_0^{-\theta}\Phi(b^{[N_0]}) \\ &\le 2^{1/t} C_4 N_0^{-\theta}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}). \end{aligned}$$

Combining this with (6.19) and (6.10), we have

$$\begin{aligned} \Phi(\{(T^{(2)}(k,j))^{1/t}\}_{(k,j)\in\mathcal{I}_q}) &\le (4C_3)^{1/t} C_4 N_0^{-\theta}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) \\ &\le \frac{\epsilon}{2M_1 C_5^{1/t}}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}). \end{aligned}$$

This proves (6.17) and completes the proof of the lemma. □

Lemma 6.8. *There exists a constant $C_{6.8}$ which depends only on the complex dimension n such that the inequality*

$$H_{t,i}(g;k,j) \le C_{6.8} 2^{2ni} J(g;k,j)$$

holds for all $g \in L^2(S, d\sigma)$, $(k,j) \in I$, $i \in \mathbf{Z}_+$ and $1 \le t < \infty$.

Proof. Let $g \in L^2(S, d\sigma)$ and $(k,j) \in I$. If $\zeta \in B_{k,j}$ and $i \in \mathbf{Z}_+$, then $B(\zeta, 2^{-k-i-2}) \subset C_{k,j}$, and consequently

$$\begin{aligned}
|(R_{k+i}g)(\zeta) - g_{C_{k,j}}| &\le \frac{1}{\sigma(B(\zeta, 2^{-k-i-2}))} \int_{B(\zeta,2^{-k-i-2})} |g - g_{C_{k,j}}| d\sigma \\
&\le \frac{\sigma(C_{k,j})}{\sigma(B(\zeta, 2^{-k-i-2}))} \cdot \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma \\
&\le (2^{10n}/c_1) 2^{2ni} J(g;k,j),
\end{aligned}$$

where the third $\le$ follows from (6.7). On the other hand,

$$\begin{aligned}
|g_{C_{k,j}} - g_{B_{k,j}}| &\le \frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |g_{C_{k,j}} - g| d\sigma \\
&\le \frac{\sigma(C_{k,j})}{\sigma(B_{k,j})} \cdot \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g_{C_{k,j}} - g| d\sigma \le (2^{2n}/c_1) J(g;k,j),
\end{aligned}$$

where the last $\le$ again follows from (6.7). Write $C_{6.8} = (2^{10n}/c_1) + (2^{2n}/c_1)$. Then the above two inequalities together give us

$$|(R_{k+i}g)(\zeta) - g_{B_{k,j}}| \le C_{6.8} 2^{2ni} J(g;k,j)$$

for every $\zeta \in B_{k,j}$. Recalling Definition 6.4(c), the lemma follows. □

Definition 6.9. (a) For each $(k,j) \in I$, we set

$$\begin{aligned}
E(k,j) &= \{(k',j') \in I : k' \ge k, d(u_{k',j'}, u_{k,j}) < 2^{-k+5}\} \quad \text{and} \\
G(k,j) &= \{(k',j') \in I : k' > k, A_{k',j'} \cap B_{k,j} \neq \emptyset\}.
\end{aligned}$$

(b) For $g \in L^2(S, d\sigma)$ and $(k,j) \in I$, we set

$$M(g;k,j) = \sup\{J(g;k',j') : (k',j') \in E(k,j)\}.$$

Proposition 6.10. *Let $1 \le t < \infty$. Then there exists a constant $C_{6.10} = C_{6.10}(t,n)$ such that the inequality*

$$J_t(g;k,j) \le C_{6.10} M(g;k,j)$$

holds for all $g \in L^2(S, d\sigma)$ and $(k,j) \in I$.

Obviously, Proposition 6.10 follows from a more structured version of the well-known John-Nirenberg theorem, a version that incorporates our particular decomposition scheme (3.3), (3.4) and (6.2). As such, the proof of Proposition 6.10 is relegated to the Appendix at the end of the paper.

Proposition 6.11. *Let $1 \le t < \infty$. There exists a constant $C_{6.11} = C_{6.11}(t,n)$ such that if Φ is any symmetric gauge function, $g \in L^2(S, d\sigma)$ and $\ell \in \mathbf{Z}_+$, then*

$$\Phi(\{J_t(g;\ell,i)\}_{i=1}^{m(\ell)}) \le C_{6.11} \Phi(\{J(g;k,j)\}_{(k,j)\in I}). \tag{6.22}$$

Proof. By (3.3) and (3.2), there is a natural number L such that the inequality

$$\text{card}\{j' \in \{1, \dots, m(k)\} : d(u_{k,j'}, u_{k,j}) < 2^{-k+6}\} \le L \tag{6.23}$$

holds for every $(k, j) \in I$. Let $1 \le t < \infty$ be given. Let $g \in L^2(S, d\sigma)$ and symmetric gauge function Φ also be given. To prove (6.22), it suffices to consider the case where $\Phi(\{J(g; k, j)\}_{(k,j)\in I}) < \infty$. Note that this implies

$$\sup_{(k,j)\in I} J(g; k, j) < \infty.$$

Let $\ell \in \mathbf{Z}_+$. Then for each $i \in \{1, \dots, m(\ell)\}$, there is an $h(i) \in E(\ell, i)$ such that

$$J(g; h(i)) \ge \frac{1}{2} M(g; \ell, i).$$

Applying Proposition 6.10, we have

$$J_t(g; \ell, i) \le C_{6.10} M(g; \ell, i) \le 2C_{6.10} J(g; h(i)),$$

$i \in \{1, \dots, m(\ell)\}$. Consequently,

$$\Phi(\{J_t(g; \ell, i)\}_{i=1}^{m(\ell)}) \le 2C_{6.10} \Phi(\{J(g; h(i))\}_{i=1}^{m(\ell)}). \tag{6.24}$$

If $i, i' \in \{1, \dots, m(\ell)\}$ are such that $h(i) = h(i')$, then $E(\ell, i) \cap E(\ell, i') \ne \emptyset$, which means that there is some (k_0, j_0) such that

$$d(u_{\ell,i}, u_{k_0,j_0}) < 2^{-\ell+5} \quad \text{and} \quad d(u_{\ell,i'}, u_{k_0,j_0}) < 2^{-\ell+5}.$$

Hence if $h(i) = h(i')$, then $d(u_{\ell,i}, u_{\ell,i'}) < 2^{-\ell+6}$. Thus, by (6.23), the map

$$h : \{1, \dots, m(\ell)\} \to I$$

is at most L-to-1. Therefore it follows from Lemma 6.6 that

$$\Phi(\{J(g; h(i))\}_{i=1}^{m(\ell)}) \le L\Phi(\{J(g; k, j)\}_{(k,j)\in I}).$$

Combining this with (6.24), we see that the proposition holds for the constant $C_{6.11} = 2LC_{6.10}$. $\square$

After the extensive preparation above, here is our reverse Hölder's inequality:

Proposition 6.12. *Let Φ be a symmetric gauge function satisfying condition* (DQK), *and let $1 \le t < \infty$. Then there exists a constant $C_{6.12}$ which depends only on Φ, t and the complex dimension n such that*

$$\Phi(\{J_t(g; k, j)\}_{(k,j)\in I}) \le C_{6.12} \Phi(\{J(g; k, j)\}_{(k,j)\in I}) \tag{6.25}$$

for every $g \in L^2(S, d\sigma)$.

Proof. Given Φ and t as in the statement of the proposition, Lemma 6.7 provides a $\nu \in \mathbf{N}$ such that

$$\Phi(\{G_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) \le \frac{1}{2}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) \tag{6.26}$$

for all $g \in L^2(S, d\sigma)$ and $m \in \mathbf{N}$. By Lemma 6.8, we also have

$$\Phi(\{H_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) \le C_{6.8}2^{2n\nu}\Phi(\{J(g;k,j)\}_{(k,j)\in I_m}) \tag{6.27}$$

for all $g \in L^2(S, d\sigma)$ and $m \in \mathbf{N}$. To prove (6.25), we only need to consider $g \in L^2(S, d\sigma)$ satisfying the condition $\Phi(\{J(g;k,j)\}_{(k,j)\in I}) < \infty$. By Proposition 6.10, this implies $J_t(g;k,j) < \infty$ for every $(k,j) \in I$.

Since $I_{m+\nu} = I_m \cup \{I_{m+\nu}\backslash I_m\}$, by (3.18) we have

$$\begin{aligned}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) &\le \Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) + \Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}\backslash I_m}) \\ &\le \Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) + \sum_{\ell=m+1}^{m+\nu}\Phi(\{J_t(g;\ell,i)\}_{i=1}^{m(\ell)}).\end{aligned}$$

Applying Proposition 6.11, we obtain

$$\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_{m+\nu}}) \le \Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) + \nu C_{6.11}\Phi(\{J(g;k,j)\}_{(k,j)\in I}).$$

Substituting this in (6.26), we have

$$\Phi(\{G_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) \le \frac{1}{2}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) + \nu C_{6.11}\Phi(\{J(g;k,j)\}_{(k,j)\in I}).$$

By Definition 6.4, $J_t(g;k,j) \le G_{t,\nu}(g;k,j) + H_{t,\nu}(g;k,j)$. Thus, combining the above inequality with (6.27), we find that

$$\begin{aligned}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) &\le \Phi(\{G_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) + \Phi(\{H_{t,\nu}(g;k,j)\}_{(k,j)\in I_m}) \\ &\le \frac{1}{2}\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) + (\nu C_{6.11} + C_{6.8}2^{2n\nu})\Phi(\{J(g;k,j)\}_{(k,j)\in I}).\end{aligned}$$

Thus the obvious cancellation in the above leads to

$$\Phi(\{J_t(g;k,j)\}_{(k,j)\in I_m}) \le 2(\nu C_{6.11} + C_{6.8}2^{2n\nu})\Phi(\{J(g;k,j)\}_{(k,j)\in I}).$$

Since $m \in \mathbf{N}$ is arbitrary, recalling (1.1), we conclude that the proposition holds for the constant $C_{6.12} = 2(\nu C_{6.11} + C_{6.8}2^{2n\nu})$. $\square$

7. Upper bound

We now turn to the estimate of $\|[P,M_g]\|_p^-$. As it happens, this estimate involves a new and quite elaborate interpolation scheme. In other words, this is not the standard kind of interpolation [3]. Our estimate of $\|[P,M_g]\|_p^-$ will be realized through an interpolation between the norms $\|\cdot\|_{r'}^+$ and $\|\cdot\|_r^+$, where $r'<p<r$. What complicates the matter is that estimates of $\|\cdot\|_{r'}^+$ and $\|\cdot\|_r^+$ are themselves obtained through interpolation between *Schatten classes*. Thus the estimate of $\|[P,M_g]\|_p^-$ is really a two-stage interpolation.

As in [5,6], for each operator A we introduce the distribution function

$$N_A(s)=\operatorname{card}\{j\in\mathbf{N}:s_j(A)>s\},$$

$s>0$, where $s_1(A), s_2(A),\cdots, s_j(A),\cdots$ are the s-numbers of A. Also recall from [5,(7.1)] that we have the inequality

$$N_{A+B}(s)\le N_A(s/2)+N_B(s/2).$$

We define the measure

$$d\mu(x,y)=\frac{d\sigma(x)d\sigma(y)}{|1-\langle x,y\rangle|^{2n}}$$

on $S\times S$. For each $1<p<\infty$, let $L^p_{\text{sym}}(S\times S,d\mu)$ be the collection of functions F on $S\times S$ which are L^p with respect to $d\mu$ and which satisfy the condition

$$|F(x,y)|=|F(y,x)|,\qquad (x,y)\in S\times S.$$

For each $F\in L^p_{\text{sym}}(S\times S,d\mu)$, define T_F to be the integral operator on $L^2(S,d\sigma)$ with the kernel function

$$K_F(x,y)=\frac{F(x,y)}{(1-\langle x,y\rangle)^n}.$$

For these operators we have the following weak-type inequality:

Proposition 7.1. [6,Proposition 7.1] *Given any $2<p<\infty$, there is a constant $C_{7.1}=C_{7.1}(p,n)$ such that*

$$N_{T_F}(t)\le\frac{C_{7.1}}{t^p}\iint\frac{|F(x,y)|^p}{|1-\langle x,y\rangle|^{2n}}d\sigma(x)d\sigma(y)$$

for all $F\in L^p_{\text{sym}}(S\times S,d\mu)$ and $t>0$.

Definition 7.2. (a) A subset Y of $S\times S$ is said to be *symmetric* if for every $(x,y)\in S\times S$, we have $(x,y)\in Y$ if and only if $(y,x)\in Y$.
(b) For any $g\in L^2(S,d\sigma)$ and any measurable, symmetric subset Y of $S\times S$, we let $C(g;Y)$ denote the integral operator on $L^2(S,d\sigma)$ with the kernel function

$$\chi_Y(x,y)\frac{g(y)-g(x)}{(1-\langle x,y\rangle)^n}.$$

Definition 7.3. (a) For each $k \in \mathbf{Z}_+$, let $E_k = \{(x,y) \in S \times S : 2^{-k} \le d(x,y) < 2^{-k+1}\}$.
(b) For each $(k,j) \in I$, we set $D_{k,j} = B_{k,j} \times B_{k,j}$, where $B_{k,j}$ is defined in (6.2).
(c) For each $(k,j) \in I$, we set $R_{k,j} = D_{k,j} \cap E_k$.

We are now ready to carry out the out two-stage interpolation for $\|[P,M_g]\|_p^-$. The first interpolation is a more refined version of Proposition 7.2 in [6]:

Proposition 7.4. *Let $2 < p < t < \infty$. Then there is a constant $C_{7.4} = C_{7.4}(p,t,n)$ such that the following estimate holds: Suppose that G is a subset of I and that Y is a measurable, symmetric subset of $S \times S$ satisfying the condition*

$$Y \subset \cup_{(k,j)\in G} R_{k,j}.$$

Then $\|C(g;Y)\|_p^+ \le C_{7.4}\Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G})$ for every $g \in L^2(S,d\sigma)$.

Proof. Let $2 < p < t < \infty$. By (3.2), it is elementary that there is a constant C such that

$$2^{4nk} \iint_{D_{k,j}} |g(x) - g(y)|^t d\sigma(x)d\sigma(y) \le C J_t^t(g;k,j)$$

for all $g \in L^2(S,d\sigma)$ and $(k,j) \in I$. Let G and Y be as in the statement of the proposition. To prove the proposition, it suffices to consider $g \in L^2(S,d\sigma)$ satisfying the condition $\Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G}) < \infty$.

Let us estimate $N_{C(g;Y)}(s)$, $s > 0$. For this, we will decompose the integral operator $C(g;Y)$ in the form $C(g;Y) = A_s + B_s$ and take advantage of the inequality

$$N_{C(g;Y)}(s) \le N_{A_s}(s/2) + N_{B_s}(s/2).$$

We will then estimate $N_{A_s}(s/2)$ by Proposition 7.1 and estimate $N_{B_s}(s/2)$ by using the Hilbert-Schmidt norm $\|B_s\|_2$. But first we need to define A_s and B_s.

Let us write

$$R = 2^{1/p}\frac{p}{p-1}\Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G}). \tag{7.1}$$

Set $\mathcal{N} = \mathbf{N}$ in the case $\operatorname{card}(G) = \infty$ and set $\mathcal{N} = \{1,\dots,m\}$ in the case $\operatorname{card}(G) = m < \infty$. By Lemma 2.4, there is a bijection $\pi : \mathcal{N} \to G$ such that

$$J_t(g;\pi(i)) \le R/i^{1/p} \quad \text{for every} \quad i \in \mathcal{N}. \tag{7.2}$$

Let $G(s) = \{\pi(i) : 1 \le i < (R/s)^p\}$. We define

$$W(s) = \cup_{(k,j)\in G(s)}(Y \cap R_{k,j}) \quad \text{and} \quad F(s) = Y \backslash W(s).$$

Now we let A_s and B_s be the integral operators on $L^2(S,d\sigma)$ with the kernel functions

$$\chi_{F(s)}(x,y)\frac{g(y)-g(x)}{(1-\langle x,y\rangle)^n} \quad \text{and} \quad \chi_{W(s)}(x,y)\frac{g(y)-g(x)}{(1-\langle x,y\rangle)^n}$$

respectively. We first estimate $N_{A_s}(s/2)$.

Since $Y \subset \cup_{(k,j)\in G} R_{k,j}$ by assumption, we have $F(s) \subset \cup_{(k,j)\in G\backslash G(s)} R_{k,j}$. Hence

$$\begin{aligned}
\iint_{F(s)} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y) &\le \sum_{(k,j)\in G\backslash G(s)} \iint_{R_{k,j}} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y) \\
&\le \sum_{(k,j)\in G\backslash G(s)} 2^{4nk} \iint_{D_{k,j}} |g(y)-g(x)|^t d\sigma(x)d\sigma(y) \le C \sum_{(k,j)\in G\backslash G(s)} J_t^t(g;k,j) \\
&= C \sum_{i\ge (R/s)^p} J_t^t(g;\pi(i)) \le C \sum_{i\ge (R/s)^p} (R/i^{1/p})^t \qquad \text{(by (7.2))} \\
&\le R^t \cdot C_1(\max\{1,R/s\})^{p(1-(t/p))} = C_1 R^t(\max\{1,R/s\})^{p-t},
\end{aligned}$$

where the last $\le$ is the reason why we must require $t > p$. Since the set $F(s)$ is symmetric, we can apply Proposition 7.1 to obtain

$$\begin{aligned}
N_{A_s}(s/2) &\le C_{7.1}(2/s)^t \iint_{F(s)} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y) \\
&\le C_{7.1}(2/s)^t \cdot C_1 R^t(\max\{1,R/s\})^{p-t} \le 2^t C_1 C_{7.1} R^p s^{-p}, \qquad (7.3)
\end{aligned}$$

where the last $\le$ also uses the assumption $t > p$.

To estimate $N_{B_s}(s/2)$, note that

$$\begin{aligned}
\|B_s\|_2^2 = \iint_{W(s)} \frac{|g(y)-g(x)|^2}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y) &\le \sum_{(k,j)\in G(s)} \iint_{R_{k,j}} \frac{|g(y)-g(x)|^2}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y) \\
&\le \sum_{(k,j)\in G(s)} 2^{4nk} \iint_{D_{k,j}} |g(y)-g(x)|^2 d\sigma(x)d\sigma(y) \le C_2 \sum_{(k,j)\in G(s)} J_t^2(g;k,j),
\end{aligned}$$

where the last $\le$ follows from (3.2) and Hölder's inequality. Recalling (7.2), we have

$$\begin{aligned}
\|B_s\|_2^2 &\le C_2 \sum_{\pi(i)\in G(s)} J_t^2(g;\pi(i)) \le C_2 \sum_{1\le i<(R/s)^p} (R/i^{1/p})^2 \\
&\le C_3 R^2 \cdot (R/s)^{p(1-(2/p))} = C_3 R^p s^{-p+2}.
\end{aligned}$$

Therefore

$$N_{B_s}(s/2) \le (2/s)^2 \|B_s\|_2^2 \le 4C_3 R^p s^{-p}.$$

Combining this with (7.3) and recalling (7.1), we have

$$N_{C(g;Y)}(s) \le \{2^t C_1 C_{7.1} + 4C_3\} R^p s^{-p} = C_4\{\Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G})\}^p s^{-p}.$$

If $\nu \in \mathbf{N}$ and $a_\nu > 0$ are such that $N_T(a_\nu) < \nu$, then $s_\nu(T) \le a_\nu$. Hence it follows from the above inequality that the s-numbers of $C(g;Y)$ satisfy the condition

$$s_\nu(C(g;Y)) \le (2C_4)^{1/p} \Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G})\nu^{-1/p}$$

for every $\nu \in \mathbf{N}$. Therefore

$$\|C(g;Y)\|_p^+ \le (2C_4)^{1/p}\Phi_p^+(\{J_t(g;k,j)\}_{(k,j)\in G}).$$

This completes the proof of the proposition. □

The second stage of our interpolation requires the estimates obtained in Section 2.

Proposition 7.5. *Let $2 < p < \infty$. Then there is a constant $C_{7.5} = C_{7.5}(p,n)$ such that*

$$\|[P,M_g]\|_p^- \le C_{7.5}\Phi_p^-(\{J(g;k,j)\}_{(k,j)\in I})$$

for every $g \in L^2(S,d\sigma)$.

Proof. Given $2 < p < \infty$, we pick a t such that $p < t < \infty$. Lemma 6.2 tells us that the symmetric gauge function Φ_p^- satisfies condition (DQK). Thus, by Proposition 6.12,

$$\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}) \le C_{6.12}\Phi_p^-(\{J(g;k,j)\}_{(k,j)\in I})$$

for every $g \in L^2(S,d\sigma)$. Hence it suffices to show that there is a constant C such that

$$\|[P,M_g]\|_p^- \le C\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}) \tag{7.4}$$

for every $g \in L^2(S,d\sigma)$.

To prove (7.4), we pick r and r' such that $2 < r' < p < r < t$. Given $g \in L^2(S,d\sigma)$, let us estimate $N_{[P,M_g]}(s)$, $s > 0$. The idea is to decompose $[P,M_g]$ in the form $C(g;X_s)+C(g;Y_s)$ and take advantage of the inequality

$$N_{[P,M_g]}(s) \le N_{C(g;X_s)}(s/2) + N_{C(g;Y_s)}(s/2). \tag{7.5}$$

The sets X_s and Y_s are chosen as follows. Let Δ denote the diagonal $\{(x,x) : x \in S\}$ in $S \times S$. Then, of course, $(\sigma\times\sigma)(\Delta) = 0$. For each $s > 0$ we set

$$E(s) = \{(k,j) \in I : J_t(g;k,j) \le s\}.$$

We then define

$$X_s = \cup_{(k,j)\in E(s)} R_{k,j} \quad \text{and} \quad Y_s = (S\times S)\backslash(X_s \cup \Delta).$$

Since $2 < r < t$, it follows from Proposition 7.4 that

$$\|C(g;X_s)\|_r^+ \le C_{7.4}(r,t)\Phi_r^+(\{J_t(g;k,j)\}_{(k,j)\in E(s)}).$$

By Lemma 2.6, we have

$$N_{C(g;X_s)}(s/2) \le \left(\frac{r}{r-1}\right)^r \left(\frac{2}{s}\|C(g;X_s)\|_r^+\right)^r.$$

Setting $C_1 = \{2C_{7.4}(r,t)r/(r-1)\}^r$, from the above two inequalities we obtain

$$N_{C(g;X_s)}(s/2) \le C_1 \left(\frac{1}{s}\Phi_r^+(\{J_t(g;k,j)\}_{(k,j)\in E(s)})\right)^r. \tag{7.6}$$

By (3.4) and (6.2), we have $\cup_{j=1}^{m(k)} D_{k,j} \supset E_k$ for every $k \in \mathbf{Z}_+$. Also, it is obvious that $\cup_{k=0}^{\infty} E_k = (S \times S)\backslash\Delta$. Consequently, $\cup_{(k,j)\in I} R_{k,j} = (S \times S)\backslash\Delta$. Therefore

$$Y_s \subset \cup_{(k,j)\in I\backslash E(s)} R_{k,j}.$$

Since $2 < r' < t$, it follows from Proposition 7.4 that

$$\|C(g;Y_s)\|_{r'}^+ \le C_{7.4}(r',t)\Phi_{r'}^+(\{J_t(g;k,j)\}_{(k,j)\in I\backslash E(s)}).$$

Then another application of Lemma 2.6 gives us

$$N_{C(g;Y_s)}(s/2) \le C_2 \left(\frac{1}{s}\Phi_{r'}^+(\{J_t(g;k,j)\}_{(k,j)\in I\backslash E(s)})\right)^{r'}, \tag{7.7}$$

where $C_2 = \{2C_{7.4}(r',t)r'/(r'-1)\}^{r'}$. Note that $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$ for all $a, b \in [0,\infty)$. Thus if we write $C_3 = (\max\{C_1, C_2\})^{1/p}$, then it follows from (7.5), (7.6) and (7.7) that

$$\begin{aligned} \{N_{[P,M_g]}(s)\}^{1/p} \le\, & C_3 \left(\frac{1}{s}\Phi_r^+(\{J_t(g;k,j)\}_{(k,j)\in E(s)})\right)^{r/p} \\ & + C_3 \left(\frac{1}{s}\Phi_{r'}^+(\{J_t(g;k,j)\}_{(k,j)\in I\backslash E(s)})\right)^{r'/p}. \end{aligned} \tag{7.8}$$

Since $2 < p < r$, it follows from Proposition 2.2 that

$$\int_0^\infty \left(\frac{1}{s}\Phi_r^+(\{J_t(g;k,j)\}_{(k,j)\in E(s)})\right)^{r/p} ds \le C_{2.2}\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}).$$

Similarly, since $2 < r' < p$, Proposition 2.3 tells us that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r'}^+(\{J_t(g;k,j)\}_{(k,j)\in I\backslash E(s)})\right)^{r'/p} ds \le C_{2.3}\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}).$$

Combining the above two inequalities with (7.8), we obtain

$$\int_0^\infty \{N_{[P,M_g]}(s)\}^{1/p} ds \le C_3(C_{2.2} + C_{2.3})\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}).$$

Now an application of Lemma 2.1 gives us

$$\|[P,M_g]\|_p^- \le p\int_0^\infty \{N_{[P,M_g]}(s)\}^{1/p} ds \le pC_3(C_{2.2} + C_{2.3})\Phi_p^-(\{J_t(g;k,j)\}_{(k,j)\in I}).$$

That is, (7.4) holds for the constant $C = pC_3(C_{2.2} + C_{2.3})$. This completes the proof. $\square$

Proposition 7.5 is the essential part of the proof of the upper bound in Theorem 1.2. What remains in the proof of the upper bound is to bring $\mathrm{Var}^{1/2}(g; z)$ and Bergman lattice into the picture. But this last step has been taken care of previously:

Proposition 7.6. [6,Proposition 8.4] *Given any positive number* $0 < b < \infty$, *there is a constant* $C_{7.6}$ *which depends only on* b *and* n *such that if* Γ *is a countable subset of* $\mathbf{B}$ *with the property that* $\cup_{z\in\Gamma} D(z, b) = \mathbf{B}$, *then*

$$\Phi(\{J(g; k, j)\}_{(k,j)\in I}) \le C_{7.6}\Phi(\{\mathrm{Var}^{1/2}(g; z)\}_{z\in\Gamma})$$

for every $g \in L^2(S, d\sigma)$ *and every symmetric gauge function* Φ.

Proof of the upper bound in Theorem 1.2. Given an $f \in L^2(S, d\sigma)$, write $g = f - Pf$. Then $H_f = H_g$. Let $2 < p < \infty$ and $b > 0$. Let Γ be a countable subset of $\mathbf{B}$ such that $\cup_{z\in\Gamma} D(z, b) = \mathbf{B}$. Applying Propositions 7.5 and 7.6, we have

$$\begin{aligned} \|H_f\|_p^- = \|H_g\|_p^- &\le \|[P, M_g]\|_p^- \le C_{7.5}\Phi_p^-(\{J(g; k, j)\}_{(k,j)\in I}) \\ &\le C_{7.5}C_{7.6}\Phi_p^-(\{\mathrm{Var}^{1/2}(g; z)\}_{z\in\Gamma}) = C_{7.5}C_{7.6}\Phi_p^-(\{\mathrm{Var}^{1/2}(f - Pf; z)\}_{z\in\Gamma}). \end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

Appendix

The purpose of this appendix is to provide a proof of Proposition 6.10. This proof follows the general approach of the standard John-Nirenberg theorem, as can be found, e.g., in [8,Section VI.2]. Our proof begins with the introduction of maximal functions associated with our particular spherical decomposition (3.3), (3.4) and (6.2).

Let $f \in L^1(S, d\sigma)$ and $x \in S$. Then for each $k \in \mathbf{Z}_+$ we define

$$(M_k f)(x) = \max\left\{\frac{1}{\sigma(A_{k,j})}\int_{A_{k,j}} |f|d\sigma : x \in A_{k,j}, 1 \le j \le m(k)\right\} \quad \text{and}$$

$$(\tilde{M}_k f)(x) = \max\left\{\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |f|d\sigma : x \in B_{k,j}, 1 \le j \le m(k)\right\}.$$

Lemma A.1. *There exists a constant* $C_{8.1}$ *such that*

$$(\tilde{M}_{k+3} f)(x) \le C_{8.1}(M_k f)(x)$$

for all $f \in L^1(S, d\sigma)$, $x \in S$, *and* $k \in \mathbf{Z}_+$.

Proof. By (6.7), there is a constant $C_{8.1}$ such that

$$\frac{\sigma(A_{k,j})}{\sigma(B_{k+3,i})} \le C_{8.1}$$

for all $k \in \mathbf{Z}_+$, $j \in \{1, \ldots, m(k)\}$ and $i \in \{1, \ldots, m(k+3)\}$. Let $f \in L^1(S, d\sigma)$, $x \in S$, and $k \in \mathbf{Z}_+$ be given. By (3.4), there is a $j^* \in \{1, \ldots, m(k)\}$ such that $x \in B(u_{k,j^*}, 2^{-k})$. By (6.2), we have $A_{k,j^*} \supset B(x, 2^{-k})$. Again by (6.2), if $i \in \{1, \ldots, m(k+3)\}$ is such that $x \in B_{k+3,i}$, then $B(x, 2^{-k}) \supset B_{k+3,i}$. Thus

$$A_{k,j^*} \supset B_{k+3,i} \quad \text{for every} \quad i \in \{1, \ldots, m(k+3)\} \quad \text{such that} \quad x \in B_{k+3,i}.$$

Therefore if $x \in B_{k+3,i}$, then

$$\frac{1}{\sigma(B_{k+3,i})} \int_{B_{k+3,i}} |f| d\sigma \le \frac{\sigma(A_{k,j^*})}{\sigma(B_{k+3,i})} \cdot \frac{1}{\sigma(A_{k,j^*})} \int_{A_{k,j^*}} |f| d\sigma \le C_{8.1}(M_k f)(x).$$

Combining this with the definition of $(\tilde{M}_{k+3} f)(x)$, the lemma follows. □

Lemma A.2. *There exist constants $C_{8.2}$ and $C_{8.3}$ such that the following estimates hold: Suppose $f \in L^1(S, d\sigma)$, $(k, j) \in I$ and $r > 0$ satisfy the condition*

$$\frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |f| d\sigma \le r. \tag{A.1}$$

Then there exists a subset G of $G(k, j)$ (see Definition 6.9) such that
(a) $|f(x)| \le C_{8.2} r$ *for σ-a.e.* $x \in B_{k,j} \backslash \{\cup_{(\kappa,i) \in G} B_{\kappa,i}\}$;
(b)

$$\sum_{(\kappa,i) \in G} \sigma(B_{\kappa,i}) \le \frac{M_1}{r} \int_{C_{k,j}} |f| d\sigma,$$

where M_1 is the natural number in (6.4);
(c) *for every $(\kappa, i) \in G$, we have*

$$\frac{1}{\sigma(B_{\kappa,i})} \int_{B_{\kappa,i}} |f| d\sigma \le C_{8.3} r.$$

Proof. By (3.2), there is a constant $0 < C_{8.2} < \infty$ such that

$$\frac{\sigma(B(\zeta, 2^{-\kappa+5}))}{\sigma(B(\xi, 2^{-\kappa}))} \le C_{8.2}$$

for all $\zeta, \xi \in S$ and $\kappa \in \mathbf{Z}_+$. Suppose that (A.1) holds. Then define

$$B = \left\{ x \in B_{k,j} : \limsup_{\kappa \to \infty} (M_\kappa f)(x) > C_{8.2} r \right\}.$$

It follows from (3.2) that if x is a Lebesgue point for $|f|$, then

$$\lim_{\kappa \to \infty} (M_\kappa f)(x) = |f(x)|.$$

Hence $|f(x)| \le C_{8.2}r$ for σ-a.e. $x \in B_{k,j} \backslash B$. Consequently, it suffices to find a subset G of $G(k,j)$ such that

$$\cup_{(\kappa,i)\in G} B_{\kappa,i} \supset B \tag{A.2}$$

and such that estimates (b) and (c) hold. To find such a G, we first recall that if $\kappa \ge k$ and if $A_{\kappa,i} \cap B_{k,j} \neq \emptyset$, then $A_{\kappa,i} \subset C_{k,j}$.

Let $x \in B_{k,j}$. For any $1 \le \nu \le 3$, if $i \in \{1, \dots, m(k+\nu)\}$ is such that $x \in A_{k+\nu,i}$, then

$$\frac{1}{\sigma(A_{k+\nu,i})} \int_{A_{k+\nu,i}} |f| d\sigma \le \frac{\sigma(C_{k,j})}{\sigma(A_{k+\nu,i})} \cdot \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |f| d\sigma \le C_{8.2}r.$$

This shows that $(M_{k+\nu}f)(x) \le C_{8.2}r$ for all $x \in B_{k,j}$ and $\nu = 1, 2, 3$. Thus for each $x \in B$, there is a natural number $\kappa(x) > k+3$ such that

$$(M_{\kappa(x)}f)(x) > C_{8.2}r \quad \text{and} \quad (M_{\kappa(x)-3}f)(x) \le C_{8.2}r.$$

Set $C_{8.3} = C_{8.1}C_{8.2}$. By Lemma A.1, the second inequality above implies

$$(\tilde{M}_{\kappa(x)}f)(x) \le C_{8.3}r. \tag{A.3}$$

For each $x \in B$, there is an $i(x) \in \{1, \dots, m(\kappa(x))\}$ such that $x \in A_{\kappa(x),i(x)}$ and

$$\frac{1}{\sigma(A_{\kappa(x),i(x)})} \int_{A_{\kappa(x),i(x)}} |f| d\sigma = (M_{\kappa(x)}f)(x) > C_{8.2}r. \tag{A.4}$$

Let

$$\mathcal{G} = \{(\kappa(x), i(x)) : x \in B\}.$$

Then, of course, $\mathcal{G} \subset G(k,j)$ and $\cup_{(\kappa,i)\in\mathcal{G}} A_{\kappa,i} \supset B$. Our desired set G will be a subset of $\mathcal{G}$, defined as follows. Recall that $\kappa(x) \ge k+4$ for every $x \in B$. We define

$$G_{k+4} = \{(\kappa(x), i(x)) : x \in B \;\text{ and }\; \kappa(x) = k+4\}.$$

Inductively, suppose that $\ell \ge 4$ and that we have defined G_{k+q} for every $4 \le q \le \ell$. Then we define

$$G_{k+\ell+1} = \{(\kappa(x), i(x)) : x \in B, \kappa(x) = k+\ell+1 \text{ and } A_{\kappa(x),i(x)} \cap \{\cup_{q=4}^{\ell} \cup_{(\kappa,i)\in G_{k+q}} A_{\kappa,i}\} = \emptyset\}.$$

This defines $G_{k+\ell}$ for every $\ell \ge 4$. Let

$$G = \cup_{\ell=4}^{\infty} G_{k+\ell}.$$

Let us verify that G has the desired properties. First of all, by the above inductive process, if $x \in B$ is such that $(\kappa(x), i(x)) \notin G$, then there is a $(\kappa, i) \in G$ with $\kappa < \kappa(x)$ such that $A_{\kappa(x),i(x)} \cap A_{\kappa,i} \neq \emptyset$. Since $\kappa < \kappa(x)$, this implies $A_{\kappa(x),i(x)} \subset B_{\kappa,i}$. Hence (A.2) holds.

To verify (b), for each $\ell \geq 4$ we define

$$\Delta_\ell = \bigcup_{(k+\ell,i)\in G_{k+\ell}} A_{k+\ell,i}.$$

It follows from (6.4) that

$$\sum_{(k+\ell,i)\in G_{k+\ell}} \chi_{A_{k+\ell,i}} \leq M_1 \chi_{\Delta_\ell}$$

for every $\ell \geq 4$. By (A.4), we have

$$C_{8.2}\sigma(A_{k+\ell,i}) < \frac{1}{r}\int_{A_{k+\ell,i}} |f| d\sigma$$

for every $(k+\ell, i) \in G_{k+\ell}$. Combining the above two inequalities, we have

$$\sum_{(k+\ell,i)\in G_{k+\ell}} C_{8.2}\sigma(A_{k+\ell,i}) < \frac{1}{r}\sum_{(k+\ell,i)\in G_{k+\ell}} \int_{A_{k+\ell,i}} |f| d\sigma \leq \frac{M_1}{r}\int_{\Delta_\ell} |f| d\sigma.$$

Since $C_{8.2}\sigma(A_{k+\ell,i}) \geq \sigma(B_{k+\ell,i})$ for every $(k+\ell, i) \in G_{k+\ell}$, we obtain

$$\sum_{(k+\ell,i)\in G_{k+\ell}} \sigma(B_{k+\ell,i}) \leq \frac{M_1}{r}\int_{\Delta_\ell} |f| d\sigma,$$

$\ell \geq 4$. If $(k+\ell, i) \in G_{k+\ell}$, then $A_{k+\ell,i} \cap B \neq \emptyset$. Hence $\Delta_\ell \subset C_{k,j}$ for every $\ell \geq 4$. The definition of the $G_{k+\ell}$'s ensures that $\Delta_\ell \cap \Delta_{\ell'} = \emptyset$ for all $4 \leq \ell < \ell'$. Therefore

$$\sum_{(\kappa,i)\in G} \sigma(B_{\kappa,i}) = \sum_{\ell=4}^{\infty} \sum_{(k+\ell,i)\in G_{k+\ell}} \sigma(B_{k+\ell,i}) \leq \frac{M_1}{r}\sum_{\ell=4}^{\infty}\int_{\Delta_\ell} |f| d\sigma \leq \frac{M_1}{r}\int_{C_{k,j}} |f| d\sigma,$$

proving (b). Finally, (c) follows simply from (A.3). Indeed for each $x \in B$, we have

$$\frac{1}{\sigma(B_{\kappa(x),i(x)})}\int_{B_{\kappa(x),i(x)}} |f| d\sigma \leq (\tilde{M}_{\kappa(x)} f)(x) \leq C_{8.3} r.$$

This completes the proof. □

Proposition A.3. *There exists a constant $C_{8.4}$ such that if $g \in L^2(S, d\sigma)$ and $(k, j) \in I$ satisfy the condition $0 < M(g; k, j) < \infty$ and if $s > 0$, then*

$$\frac{\sigma(\{x \in B_{k,j} : |g(x) - g_{B_{k,j}}| > s\})}{\sigma(B_{k,j})} \leq 2\exp\left(\frac{-s}{C_{8.4} M(g; k, j)}\right). \tag{A.5}$$

Proof. By (3.2), there is a constant C_1 such that

$$\sigma(C_{\kappa,i}) \leq C_1 \sigma(B_{\kappa,i})$$

for every $(\kappa, i) \in I$. It is easy to see that

$$|\varphi_{B_{\kappa,i}} - \varphi_{C_{\kappa,i}}| \le C_1 J(\varphi; \kappa, i)$$

for all $\varphi \in L^2(S, d\sigma)$ and $(\kappa, i) \in I$. By the homogeneity of (A.5), it suffices to consider the case where $g \in L^2(S, d\sigma)$ and $(k, j) \in I$ satisfy the condition $M(g; k, j) = 1$. Note that

$$\frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{B_{k,j}}| d\sigma \le \frac{1}{\sigma(C_{k,j})} \int_{C_{k,j}} |g - g_{C_{k,j}}| d\sigma + |g_{C_{k,j}} - g_{B_{k,j}}| \le 1 + C_1.$$

Now we apply Lemma A.2 to the pair of $f = |g - g_{B_{k,j}}|$ and (k, j), and to the number

$$r = 2C_1 M_1 (1 + C_1), \tag{A.6}$$

where M_1 is the natural number that appears in (6.4). By Lemma A.2, there is a subset $G^{(1)}$ of $G(k, j)$ such that

$$|g(x) - g_{B_{k,j}}| \le C_{8.2} r \quad \text{for } \ \sigma\text{-a.e. } \ x \in B_{k,j} \backslash \{\cup_{(\kappa,i) \in G^{(1)}} B_{\kappa,i}\},$$

$$\sum_{(\kappa,i) \in G^{(1)}} \sigma(B_{\kappa,i}) \le \frac{M_1}{r} \int_{C_{k,j}} |g - g_{B_{k,j}}| d\sigma \le \frac{M_1}{r}(1 + C_1)\sigma(C_{k,j}) \le \frac{1}{2}\sigma(B_{k,j}),$$

and

$$\frac{1}{\sigma(B_{\kappa,i})} \int_{B_{\kappa,i}} |g - g_{B_{k,j}}| d\sigma \le C_{8.3} r \quad \text{for every } \ (\kappa, i) \in G^{(1)}.$$

This last inequality implies that

$$|g_{B_{\kappa,i}} - g_{B_{k,j}}| \le C_{8.3} r \quad \text{for every } \ (\kappa, i) \in G^{(1)}.$$

Also, since $G^{(1)} \subset G(k, j)$, for every $(\kappa, i) \in G^{(1)}$ we have $\kappa \ge k + 1$ and

$$d(u_{\kappa,i}, u_{k,j}) < 2 \cdot 2^{-k+2} = 2^{-1} \cdot 2^{-k+4}.$$

Inductively, suppose that $\ell \ge 1$ and that we have a subset $G^{(\ell)}$ of $\{(\kappa, i) \in I : \kappa \ge k + \ell\}$ such that

$$|g(x) - g_{B_{k,j}}| \le C_{8.2} r + (\ell - 1) C_{8.3} r \quad \text{for } \ \sigma\text{-a.e. } \ x \in B_{k,j} \backslash \{\cup_{(\kappa,i) \in G^{(\ell)}} B_{\kappa,i}\}, \tag{A.7}$$

$$\sum_{(\kappa,i) \in G^{(\ell)}} \sigma(B_{\kappa,i}) \le \frac{1}{2^\ell} \sigma(B_{k,j}), \tag{A.8}$$

and

$$|g_{B_{\kappa,i}} - g_{B_{k,j}}| \le \ell C_{8.3} r \quad \text{and} \quad d(u_{\kappa,i}, u_{k,j}) < (2^{-1} + \cdots + 2^{-\ell}) \cdot 2^{-k+4} \tag{A.9}$$

for every $(\kappa, i) \in G^{(\ell)}$. This last condition means $G^{(\ell)} \subset E(k,j)$ (see Definition 6.9), which together with the condition $M(g;k,j) = 1$ implies

$$\frac{1}{\sigma(C_{\kappa,i})}\int_{C_{\kappa,i}} |g - g_{B_{\kappa,i}}|d\sigma \le \frac{1}{\sigma(C_{\kappa,i})}\int_{C_{\kappa,i}} |g - g_{C_{\kappa,i}}|d\sigma + |g_{C_{\kappa,i}} - g_{B_{\kappa,i}}| \le 1 + C_1$$

for every $(\kappa, i) \in G^{(\ell)}$. Thus the above argument can be repeated. That is, we apply Lemma A.2 to each triple of $(\kappa, i) \in G^{(\ell)}$, $f_{\kappa,i} = |g - g_{B_{\kappa,i}}|$, and the same r given by (A.6). This gives us a subset $G^{(\ell+1)}_{\kappa,i}$ of $G(\kappa, i)$ for each $(\kappa, i) \in G^{(\ell)}$. We set

$$G^{(\ell+1)} = \bigcup_{(\kappa,i)\in G^{(\ell)}} G^{(\ell)}_{\kappa,i}.$$

By Lemma A.2(a) and (A.9),

$$|g(x) - g_{B_{k,j}}| \le C_{8.2}r + \ell C_{8.3}r$$

for σ-a.e. $x \in \{\cup_{(\kappa,i)\in G^{(\ell)}} B_{\kappa,i}\}\backslash\{\cup_{(\kappa,i)\in G^{(\ell+1)}} B_{\kappa,i}\}$. Combining this with (A.7), we have

$$|g(x) - g_{B_{k,j}}| \le C_{8.2}r + \ell C_{8.3}r \quad \text{for} \quad \sigma\text{-a.e.} \quad x \in B_{k,j}\backslash\{\cup_{(\kappa,i)\in G^{(\ell+1)}} B_{\kappa,i}\}.$$

Also,

$$\sum_{(\kappa,i)\in G^{(\ell+1)}} \sigma(B_{\kappa,i}) \le \sum_{(\kappa,i)\in G^{(\ell)}} \frac{1}{2}\sigma(B_{\kappa,i}) \le \frac{1}{2^{\ell+1}}\sigma(B_{k,j})$$

and

$$|g_{B_{\kappa,i}} - g_{B_{k,j}}| \le (\ell+1)C_{8.3}r \quad \text{for every} \quad (\kappa, i) \in G^{(\ell+1)}.$$

Furthermore, if $(\kappa, i) \in G^{(\ell+1)}$, then there is a $(\kappa', i') \in G^{(\ell)}$ such that $(\kappa, i) \in G(\kappa', i')$. Since $\kappa' \ge k + \ell$, this implies $d(u_{\kappa,i}, u_{\kappa',i'}) < 2^{-1} \cdot 2^{-\kappa'+4} \le 2^{-\ell-1} \cdot 2^{-k+4}$. By the triangle inequality,

$$d(u_{\kappa,i}, u_{k,j}) < (2^{-1} + \cdots + 2^{-\ell} + 2^{-\ell-1}) \cdot 2^{-k+4} \quad \text{for every} \quad (\kappa, i) \in G^{(\ell+1)}.$$

This completes the inductive selection of the sets $G^{(1)}, G^{(2)}, \dots, G^{(\ell)}, \dots$.

To complete the proof of the proposition, let us write $C = \max\{C_{8.2}, C_{8.3}\}r$, where, as we recall, r is fixed in (A.6). Suppose that $s \ge C$. Then there is an $\ell \in \mathbf{N}$ such that

$$\ell C \le s < (\ell + 1)C.$$

By (A.7) and (A.8), we have

$$\frac{\sigma(\{x \in B_{k,j} : |g(x) - g_{B_{k,j}}| > s\})}{\sigma(B_{k,j})} \le \frac{1}{2^\ell} = 2e^{-(\ell+1)\log 2} \le 2\exp\left(-\frac{\log 2}{C}s\right).$$

On the other hand, if $0 < s < C$, then

$$2\exp\left(-\frac{\log 2}{C}s\right) \geq 2\exp\left(-\frac{\log 2}{C}C\right) = 1 \geq \frac{\sigma(\{x \in B_{k,j} : |g(x) - g_{B_{k,j}}| > s\})}{\sigma(B_{k,j})}.$$

Hence the proposition holds for the constant $C_{8.4} = C/\log 2$. $\square$

Proof of Proposition 6.10. For any $1 \leq t < \infty$, $g \in L^2(S, d\sigma)$ and $(k,j) \in I$, we have

$$J_t^t(g;k,j) = \frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |g - g_{B_{k,j}}|^t d\sigma = t\int_0^\infty s^{t-1}\frac{\sigma(\{x \in B_{k,j} : |g(x) - g_{B_{k,j}}| > s\})}{\sigma(B_{k,j})}ds.$$

Applying Proposition A.3 to the fraction in the last integral and making the obvious substitution, we obtain

$$J_t^t(g;k,j) \leq 2t(C_{8.4}M(g;k,j))^t \int_0^\infty u^{t-1}e^{-u}du.$$

Thus Proposition 6.10 holds for the constant

$$C_{6.10} = (2t)^{1/t}C_{8.4}\left(\int_0^\infty u^{t-1}e^{-u}du\right)^{1/t}.$$

This completes the proof. $\square$

References

1. J. Arazy, S. Fisher and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), 989-1054.

2. D. Békollé, C. Berger, L. Coburn and K. Zhu, BMO *in the Bergman metric on bounded symmetric domains*, J. Funct. Anal. **93** (1990), 310-350.

3. J. Bergh and J. Löström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.

4. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611-635.

5. Q. Fang and J. Xia, *Schatten class membership of Hankel operators on the unit sphere*, J. Funct. Anal. **257** (2009), 3082-3134.

6. Q. Fang and J. Xia, *A local inequality for Hankel operators on the sphere and its application*, J. Funct. Anal. **266** (2014), 876-930.

7. M. Feldman and R. Rochberg, *Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group*, Analysis and partial differential equations, 121-159, Lecture Notes in Pure and Appl. Math., **122**, Dekker, New York, 1990.

8. J. Garnett, Bounded Analytic Functions, Academic Press, New York, London, 1981.

9. I. Gohberg and M. Krein, *Introduction to the theorey of linear nonselfadjoint operators*, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.

10. H. Li, *Hankel operators on the Bergman spaces of strongly pseudoconvex domains*, Integral Equations Operator Theory **19** (1994), 458-476.

11. H. Li and D. Luecking, *Schatten class of Hankel operators and Toeplitz operators on the Bergman space of strongly pseudoconvex domains*, in "Contemporary Mathematics," Vol. 185 (Multivariable Operator Theory, R. Curto et al eds.), Amer. Math. Soc., Providence, RI, 1995.
12. D. Luecking, *Characterization of certain classes of Hankel operators on the Bergman spaces of the unit disc*, J. Funct. Anal. **110** (1992), 247-271.
13. V. Peller, *Hankel operators of class S_p and their applications* (*rational approximation, Gaussian processes, the problem of majorization of operators*), Math. USSR Sbornik **41** (1982), 443-479.
14. V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
15. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
16. J. Xia, *Singular integral operators and norm ideals satisfying a quantitative variant of Kuroda's condition*, J. Funct. Anal. **228** (2005), 369-393.
17. J. Xia, *Boundedness and compactness of Hankel operators on the sphere*, J. Funct. Anal. **255** (2008), 25-45.
18. J. Xia, *Bergman commutators and norm ideals*, J. Funct. Anal. **263** (2012), 988-1039.
19. K. Zhu, *Operator theory in function spaces.* Second edition. Mathematical Surveys and Monographs, **138**, American Mathematical Society, Providence, RI, 2007.

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