6 Multilevel mathematical programming

Sequential optimization problems arise frequently in many fields, including economics, operations research, statistics and control theory. The theory and its applications have appeared in many scientific disciplines.

6.1 Problem definition

Let the decision variable space (Euclidean n-space), $\mathbb{R}^n \ni x = (x_1, x_2, \ldots, x_n)$, be partitioned among r levels,

$$\mathbb{R}^{n_k} \ni x^k = (x_1^k, x_2^k, \ldots, x_{n_k}^k) \quad \text{for } k = 1, \ldots, r,$$

where $\sum_{k=1}^{r} n_k = n$.

Denote the maximization of a function $f(x)$ over $\mathbb{R}^n$ by varying only $x^k \in \mathbb{R}^{n_k}$ given fixed $x^{k+1}, x^{k+2}, \ldots, x^r$ in $\mathbb{R}^{n_{k+1}} \times \mathbb{R}^{n_{k+2}} \times \cdots \times \mathbb{R}^{n_r}$ by

$$\max\{f(x) \,:\, (x^k \,|\, x^{k+1}, x^{k+2}, \ldots, x^r)\}. \tag{1}$$

The level one problem:

$$(P^1) \begin{cases} \max & \{f_1(x) \,:\, (x^1 \,|\, x^2, \ldots, x^r)\} \\ \text{st:} & x \in S^1 = S \end{cases}$$

The feasible region, $S = S^1$, is defined as the **level-one feasible region.** The solutions to P^1 in $\mathbb{R}^n_1$ for each fixed $x^2, x^3, \ldots, x^r$ form a set,

$$S^2 = \{\hat{x} \in S^1 \,:\, f_1(\hat{x}) = \max\{f_1(x) \,:\, (x^1 \,|\, \hat{x}^2, \hat{x}^3, \ldots, \hat{x}^r)\}\},$$

called the **level-two feasible region** over which $f_2(x)$ is then maximized by varying x^2 for fixed $x^3, x^4, \ldots, x^r$.

Thus the problem at level two is given by

$$(P^2)\begin{cases} \max & \{f_2(x) \,:\, (x^2 \,|\, x^3, x^4, \ldots, x^r)\} \\ \text{st:} & x \in S^2 \end{cases}$$

In general, the **level-k feasible region** is defined as

$$S^k = \{\hat{x} \in S^{k-1} \,|\, f_{k-1}(\hat{x}) = \max\{f_{k-1}(x) \,:\, (x^{k-1} \,|\, \hat{x}^k, \ldots, \hat{x}^r)\}\},$$

The problem at each level is

$$(P^k)\begin{cases} \max & \{f_k(x) \,:\, (x^k \,|\, x^{k+1}, \ldots, x^r)\} \\ \text{st:} & x \in S^k \end{cases}$$

which is a function of $x^{k+1}, \ldots, x^r$, and

$$(P^r) : \max_{x \in S^r} f_r(x)$$

defines the entire problem.

This establishes a collection of nested mathematical programming problems $\{P^1, \ldots, P^r\}$.

Question 6.1 P^k depends on given $x^{k+1}, \ldots, x^r$, and only x^k is varied. But $f^k(x)$ is defined over all $x^1, \ldots, x^r$. Where are the variables $x^1, \ldots, x^{k-1}$ in problem P^k? ■

Note that the objective at level k, $f_k(x)$, is defined over the decision space of all levels. Thus, the level-k planner may have his objective function determined, in part, by variables controlled at other levels. However, by controlling x^k, after decisions from levels $k+1$ to r have been made, level k may influence the policies at level $k-1$ and hence all lower levels to improve his own objective function.

6.2 A more general definition

$x \in \mathbb{R}^N$ partitioned as (x^a, x^b).

For closed and bounded region $S \subset \mathbb{R}^N$ define:

$$\Psi_f(S) = \{\hat{x} \in S \,:\, f(\hat{x}) = \max\{f(x) \,|\, (x^a \,|\, \hat{x}^b)\}\}$$

as the **set of rational reactions** of f over S.

Sometimes called the *inducible region.*

If for a fixed $\hat{x}^b$ there exists a unique $\hat{x}^a$ that maximizes $f(x^a, \hat{x}^b)$ over $(x^a, \hat{x}^b) \in S$, then there induced a mapping

$$\hat{x}^a = \psi_f(\hat{x}^b)$$

Then

$$\Psi_f(S) = S \cap \{(x^a, x^b) : x^a = \psi_f(x^b)\}$$

If $S = S^1$ is the level-one feasible region, the level-two feasible region is

$$S^2 = \Psi_{f_1}(S^1)$$

and the level-k feasible region is

$$S^k = \Psi_{f_{k-1}}(S^{k-1})$$

Note 6.1 Even if S^1 is convex, $S^k = \Psi_{f_{k-1}}(S^{k-1})$ for $k \geq 2$ are typically non-convex sets. ■

6.3 The two-level linear resource control problem

The two-level linear resource control problem is the multilevel programming problem of the form

$$\begin{aligned} \max \quad & c^2 x \\ \text{st:} \quad & x \in S^2 \end{aligned}$$

where

$$S^2 = \{\hat{x} \in S^1 : c^1\hat{x} = \max\{c^1 x : (x^1 \,|\, \hat{x}^2)\}\}$$

and

$$S^1 = S = \{x : A^1 x^1 + A^2 x^2 \leq b,\ x \geq 0\}$$

Here, level 2 controls x^2 which, in turn, varies the resource space of level one by restricting $A^1x^1 \leq b - A^2x^2$.

The nested optimization problem can be written as:

$$(P^2)\begin{cases} \max & \{c^2x = c^{21}x^1 + c^{22}x^2 \ : \ (x^2)\} \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{c^1x = c^{11}x^1 + c^{12}x^2 \ : \ (x^1 \,|\, x^2)\} \\ \text{st:} & A^1x^1 + A^2x^2 \leq b \\ & x \geq 0 \end{cases} \end{cases}$$

Question 6.2 Suppose someone gives you a proposed solution x^* to problem P^2. Develop an "easy" way to test that x^* is, in fact, the solution to P^2. ■

Question 6.3 What is the solution to P^2 if $c^1 = c^2$. What happens if c^1 is substituted with $\alpha c^1 + (1 - \alpha)c^2$ for some $0 \leq \alpha \leq 1$? ■

6.4 The two-level linear price control problem

The two-level linear price control problem is another special case of the general multilevel programming problem. In this problem, level two controls the cost coefficients of level one:

$$(P^2)\begin{cases} \max & \{c^2x = c^{21}x^1 + c^{22}x^2 \,:\, (x^2)\} \\ \text{st:} & A^2x^2 \leq b^2 \\ & \text{where } x^1 \text{ solves} \\ & (P^1)\begin{cases} \max & \{(x^2)^{\mathsf{t}}x^1 \,:\, (x^1 \mid x^2)\} \\ \text{st:} & A^1x^1 \leq b^1 \\ & x^1 \geq 0 \end{cases} \end{cases}$$

In this problem, level two controls the cost coefficients of level one.

6.5 Properties of S^2

Theorem 6.1 *Suppose $S^1 = \{x \,:\, Ax = b,\ x \geq 0\}$ is bounded. Let*

$$S^2 = \{\hat{x} = (\hat{x}^1, \hat{x}^2) \in S^1 \,:\, c^1\hat{x}^1 = \max\{c^1x^1 \,:\, (x^1 \mid \hat{x}^2)\}\}$$

then the following hold:

- *(i)* $S^2 \subseteq S^1$
- *(ii) Let $\{y_t\}_{t=1}^{\ell}$ be any ℓ points of S^1, such that $x = \sum_t \lambda_{t=1}^{\ell} y_t \in S^2$ with $\lambda_t \geq 0$ and $\sum_t \lambda_t = 1$. Then $\lambda_t > 0$ implies $y_t \in S^2$.*

Proof: See Bialas and Karwan [4].

A set $S^2 \subseteq S^1$ with the above properties is called a **shaving** of S^1.

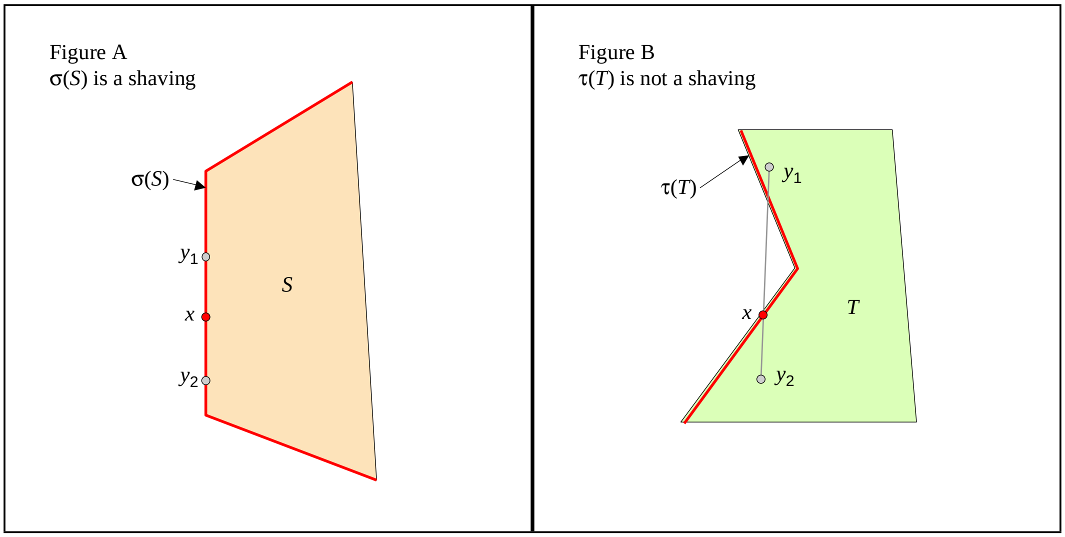
Note 6.2 The following results are due to Wen [19] (Chapter 2).

- shavings of shavings are shavings.
- shavings can be decomposed into convex sets that are shavings
- a convex set is always a shaving of itself.
- a relationship between shavings and the Karush-Kuhn-Tucker conditions for linear programming problems.

■

Definition 6.1 *Let $S \subseteq \mathbb{R}^n$. A set $\sigma(S) \subseteq S$ is a* shaving *of S if and only if for any $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S)$, the statement $\{\lambda_i > 0\}$ implies $y_i \in \sigma(S)$.*

The following figures illustrate the notion of a shaving.



The red region, $\sigma(S)$, in Figure A is a shaving of the set S. However in Figure B, the point $\lambda_1 y_1 + \lambda_2 y_2 = x \in \tau(T)$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 > 0$, $\lambda_2 > 0$. But y_1 and y_2 do not belong to $\tau(T)$. Hence $\tau(T)$ is not a shaving.

Theorem 6.2 *Suppose $T = \sigma(S)$ is a shaving of S and $\tau(T)$ is a shaving of T. Let $\tau \circ \sigma$ denote the composition of the functions τ and σ. Then $\tau \circ \sigma(S)$ is a shaving of S.*

Proof: Let $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0, \ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S) = T$.

Suppose $\lambda_i > 0$. Since $\sigma(S)$ is a shaving of S then $y_i \in \sigma(S) = T$. Since $\tau(T)$ is a shaving of T, $y_i \in T$, and $\lambda_i > 0$ then $y_i \in \tau(T)$. Therefore $y_i \in \tau(\sigma(S))$ so $\tau \circ \sigma(S)$ is a shaving of S. ■

It is easy to prove the following theorem:

Theorem 6.3 *If S is a convex set, the $\sigma(S) = S$ is a shaving of S.*

Theorem 6.4 *Let $S \subseteq \mathbb{R}^N$. Let $\sigma(S)$ be a shaving of S. If x is an extreme point of $\sigma(S)$, then x is an extreme point of S.*

Proof: See Bialas and Karwan [4].

Corollary 6.1 *An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables (S^1).*

Proof: See Bialas and Karwan [4].

These results were generalized to n-levels by Wen [19]. Using Theorems 6.2 and 6.4, if f_k is linear and S^1 is a bounded convex polyhedron then the extreme points of

$$S^k = \Psi_{k-1}\Psi_{k-2}\cdots\Psi_2\Psi_1(S^1)$$

are extreme points of S^1. This justifies the use of extreme point search procedures to finding the solution to the n-level linear resource control problem.

6.6 Cooperative Stackelberg games

The multilevel programming problem is actually a Stackelberg game. Suppose we allowed payers in that game to form coalitions?

- which coalitions will tend to form,
- are the coalitions enforceable, and
- what will be the resulting distribution of wealth to each of the players?

Games in partition function form (see Lucas and Thrall [16] and Shenoy [17]) provides a framework for answering these questions.

Definition 6.2 *An* **abstract game** *is a pair* (X, dom) *where* X *is a set whose members are called* **outcomes** *and* dom *is a binary relation on* X *called* **domination***.*

Let $G = \{1, 2, \dots, n\}$ denote the set of n players.

Let $\mathcal{P} = \{R_1, R_2, \dots, R_M\}$ denote a coalition structure where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{M} R_i = G$.

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \dots, \{n\}\}$ denote the coalition structure where no coalitions have formed.

Let $\mathcal{P}_G \equiv \{G\}$ denote the **grand coalition**.

Assume that utility is additive and transferable.

Suppose $x = (x^1, \dots, x^n)$ is the vector of strategies for players $1, 2, \dots, n$.

Under coalition structure $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$ the value of coalition R_j is,

$$f'_{R_j}(x) = \sum_{i \in R_j} f_i(x).$$

Hence instead of maximizing $f_i(x)$, player $i \in R_j$ will now be maximizing $f'_{R_j}(x)$.

Let $\hat{x}(\mathcal{P})$ denote the solution to the resulting n-level optimization problem.

This is the cooperative Stackelberg strategy under coalition structure $\mathcal{P}$.

Definition 6.3 *Suppose that S^1 is compact and $\hat{x}(\mathcal{P})$ is unique. The value of (or payoff to) coalition $R_j \in \mathcal{P}$, denoted by $v(R_j, \mathcal{P})$, is given by*

$$v(R_j, \mathcal{P}) \equiv \sum_{i \in R_j} f_i(\hat{x}(\mathcal{P})).$$

Note 6.3 The function v need not be superadditive ■

Definition 6.4 *A* **solution configuration** *is a pair $(r, \mathcal{P})$, where r is an n-dimensional vector (called an* **imputation***) whose elements r_i $(i = 1, \ldots, n)$ represent the payoff to each player i under coalition structure $\mathcal{P}$.*

Definition 6.5 *A solution configuration $(r, \mathcal{P})$ is a* **feasible solution configuration** *if and only if $\sum_{i \in R} r_i \leq v(R, \mathcal{P})$ for all $R \in \mathcal{P}$.*

Let Θ be the set of all feasible solution configurations.

Definition 6.6 *Let* $(r, \mathcal{P}_r)$, $(s, \mathcal{P}_s) \in \Theta$. *Then* $(r, \mathcal{P}_r)$ **dominates** $(s, \mathcal{P}_s)$ *if and only if there exists an nonempty* $R \in \mathcal{P}$*, such that*

$$r_i > s_i \quad \textit{for all} \quad i \in R \quad \textit{and} \tag{2}$$

$$\sum_{i \in R} r_i \leq v(R, \mathcal{P}_r). \tag{3}$$

We write $(r, \mathcal{P}_r)\text{dom}(s, \mathcal{P}_s)$

Definition 6.7 *The* **core**, $\mathcal{C}$, *of an abstract game is the set of undominated, feasible solution configurations.*

Summary

- A model of the formation of coalitions among players in a Stackelberg game.
- Perfect information
- Coalitions are allowed to form freely.
- For every coalition structure, the order of the players' actions remains the same.
- Each coalition earns the combined proceeds that each individual coalition member would have received in the original Stackelberg game.

- A player acts for the joint benefit of the members of his coalition.

6.7 Results

Lemma 6.1 *If solution configuration* $(z, \mathcal{P}) \in \Theta$ *then*

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P}_G)) = v(G, \mathcal{P}_G) \equiv V^*.$$

Theorem 6.5 *If* $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ *then* $\sum_{i=1}^{n} z_i = V^*$*.*

Theorem 6.6 *The abstract game* (Θ, dom) *has* $\mathcal{C} = \emptyset$ *if there exists coalition structures* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ *and coalitions* $R_j \in \mathcal{P}_j$ $(j = 1, \ldots, m)$ *with* $R_j \cap R_k = \emptyset$ *for all* $j \neq k$ *such that*

$$\sum_{j=1}^{m} v(R_j, \mathcal{P}_j) > V^*. \tag{4}$$

Theorem 6.7 *If* $n = 2$ *then* $\mathcal{C} \neq \emptyset$*.*

6.8 An Example

Chew's [14] container game.

Let c_{ij} represent the reward to player i if the commodity controlled by player j is placed in the container.

Let $C = [c_{ij}]$

Let x_j = the amount of commodity j

Must have

$$\begin{aligned} \sum_{j=1}^{n} x_j &\leq 1 \\ x_j &\geq 0 \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}.$$

Note that Cx^{T} is a vector whose components represent the earnings to each player.

Chew [14] provides a simple procedure to solve this game. The algorithm requires $c_{11} > 0$.

Step 0: Initialize i=1 and j=1. Go to *Step 1.*

Step 1: If $i = n$, stop. The solution is $\hat{x}_j = 1$ and $\hat{x}_k = 0$ for $k \neq j$. If $i \neq n$, then go to *Step 2.*

Step 2: Set $i = i + 1$. If $c_{ii} > c_{ij}$, then set $j = i$. Go to *Step 1.*

If no ties occur in *Step 2* (i.e., $c_{ii} \neq c_{ij}$) then it can be shown that the above algorithm solves the problem (see Chew [14]).

Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 0 & 3 \\ 2 & 5 & 1 \end{bmatrix}.$$

Using $\mathcal{P}_0 = \{\{1\}, \{2\}, \{3\}\}$
Outcome $(x_1, x_2, x_3) = (1, 0, 0)$
$v(\{1\}, \mathcal{P}_0) = 4$
$v(\{2\}, \mathcal{P}_0) = 1$
$v(\{3\}, \mathcal{P}_0) = 2$. Using $\mathcal{P} = \{\{1\}, \{2, 3\}\}$

$$C_{\mathcal{P}} = \begin{bmatrix} 4 & 1 & 4 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

Under $\mathcal{P} = \{\{1\}, \{2, 3\}\}$
Outcome $(0, 1, 0)$
$v(\{1\}, \mathcal{P}) = 1$
$v(\{2, 3\}, \mathcal{P}) = 5$. Using $\mathcal{P}_G = \{\{1, 2, 3\}\}$

$$C_{\mathcal{P}_G} = \begin{bmatrix} 7 & 6 & 8 \\ 7 & 6 & 8 \\ 7 & 6 & 8 \end{bmatrix}$$

Under $\mathcal{P}_G = \{\{1, 2, 3\}\}$
Outcome $(0, 0, 1)$
$v(\{1, 2, 3\}, \mathcal{P}_G) = 8$.

Note that

$$v(\{1\}, \mathcal{P}_0) + v(\{2, 3\}, \mathcal{P}) > v(\{1, 2, 3\}, \mathcal{P}_G).$$

From Theorem 6.6, we know that the core for this game is empty.

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