IE675 Game Theory

Lecture Note Set 5

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5 STATIC COOPERATIVE GAMES

5.1 Some introductory examples

Consider a game with three players 1, 2 and 3. Let $N = \{1, 2, 3\}$ Suppose that the players can freely form coalitions. In this case, the possible *coalition structures* would be

$$\{\{1\}, \{2\}, \{3\}\} \quad \{\{1, 2, 3\}\}$$

$$\{\{1, 2\}, \{3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{2, 3\}, \{1\}\}$$

Once the players form their coalition(s), they inform a referee who pays each coalition an amount depending on its membership. To do this, the referee uses the function $v : 2^N \to \mathbb{R}$. Coalition S receives $v(S)$. This is a game in **characteristic function form** and v is called the **characteristic function.**

For simple games, we often specify the characteristic function without using brackets and commas. For example,

$$v(12) \equiv v(\{1, 2\}) = 100$$

The function v may actually be based on another game or an underlying decision-making problem.

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[2]Much of the material for this section has been cultivated from the lecture notes of Louis J. Billera and William F. Lucas. The errors and omissions are mine.

An important issue is the division of the game's proceeds among the players. We call the vector $(x_1, x_2, \dots, x_N)$ of these payoffs an **imputation**. In many situations, the outcome of the game can be expressed solely in terms of the resulting imputation.

Example 5.1. Here is a three-person, constant sum game:

$$
\begin{aligned}
&v(123) = 100 \\
&v(12) = v(13) = v(23) = 100 \\
&v(1) = v(2) = v(3) = 0
\end{aligned}
$$

How much will be given to each player? Consider solutions such as

$$
\begin{aligned}
(x_1, x_2, x_3) &= (\tfrac{100}{3}, \tfrac{100}{3}, \tfrac{100}{3}) \\
(x_1, x_2, x_3) &= (50, 50, 0)
\end{aligned}
$$

Example 5.2. This game is similar to Example 5.1.

$$
\begin{aligned}
&v(123) = 100 \\
&v(12) = v(13) = 100 \\
&v(23) = v(1) = v(2) = v(3) = 0
\end{aligned}
$$

Player 1 has veto power but if Player 2 and Player 3 form a coalition, they can force Player 1 to get nothing from the game. Consider this imputation as a solution:

$$
(x_1, x_2, x_3) = (\tfrac{200}{3}, \tfrac{50}{3}, \tfrac{50}{3})
$$

5.2 Cooperative games with transferable utility

Cooperative TU (transferable utility) games have the following ingredients:

1. a characteristic function $v(S)$ that gives a value to each subset $S \subset N$ of players
2. payoff vectors called *imputation* of the form $(x_1, x_2, \dots, x_n)$ which represents a realizable distribution of wealth

3. a preference relation over the set of imputations
4. solution concepts

   Global: *stable sets*

   > solutions outside of the stable set can be blocked by some coalition, and nothing in the stable set can be blocked by another member of the stable set.

   Local: *bargaining sets*

   > any objection to an element of a bargaining set has a counter-objection.

   Single point: *Shapley value*

Definition 5.1. *A* **TU game in characteristic function form** *is a pair* (N, v) *where* $N = \{1, \ldots, n\}$ *is the set of players and* $v : 2^N \to \mathbb{R}$ *is the characteristic function.*

Note 5.1. We often assume either that the game is

superadditive: $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$

or that the game is

cohesive: $v(N) \geq v(S)$ for all $S \subseteq N$

We define the set of *imputations* as

$$A(v) = \{x \mid \textstyle\sum_{i=1}^{n} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \; \forall\, i \in N\} \subset \mathbb{R}^N$$

If $S \subseteq N$, $S \neq \emptyset$ and $x, y \in A(v)$ then we say that x **dominates** y **via** S, ($x \operatorname{dom}_S y$) if and only if

1. $x_i > y_i$ for all $i \in S$
2. $\sum_{i \in S} x_i \leq v(S)$

If x dominates y via S, we write $x \operatorname{dom}_S y$.

If $x \operatorname{dom}_S y$ for some $S \subseteq N$ then we say that x **dominates** y and write $x \operatorname{dom} y$.

For $x \in A(v)$, we define the **dominion of** x **via** S as

$$\operatorname{Dom}_S x \equiv \{y \in A(v) \mid x \operatorname{dom}_S y\}$$

For any $B \subseteq A(v)$ we define

$$\mathsf{Dom}_S B \equiv \bigcup_{y \in B} \mathsf{Dom}_S\, y$$

and

$$\mathsf{Dom}\, B \equiv \bigcup_{T \subseteq N} \mathsf{Dom}_T\, B$$

We say that $K \subset A(v)$ is a **stable set** if

1. $K \cap \mathsf{Dom}\, K = \emptyset$
2. $K \cup \mathsf{Dom}\, K = A(v)$

In other words, $K = A(v) - \mathsf{Dom}\, K$

The **core** is defined as

$$\mathcal{C} \equiv \{x \in A(v) \mid \textstyle\sum_{i \in S} x_i \geq v(S)\ \forall\, S \subset N\}$$

Note 5.2. If the game is cohesive, the core is the set of undominated imputations.

Theorem 5.1. *The core of a cooperative TU game (N, v) has the following properties:*

1. *The core $\mathcal{C}$ is an intersection of half spaces.*
2. *If stable sets K_α exist, then $\mathcal{C} \subset \cap_\alpha K_\alpha$*
3. $(\cap_\alpha K_\alpha) \cap \mathsf{Dom}\mathcal{C} = \emptyset$

Note 5.3. For some games (e.g., constant sum games) the core is empty.

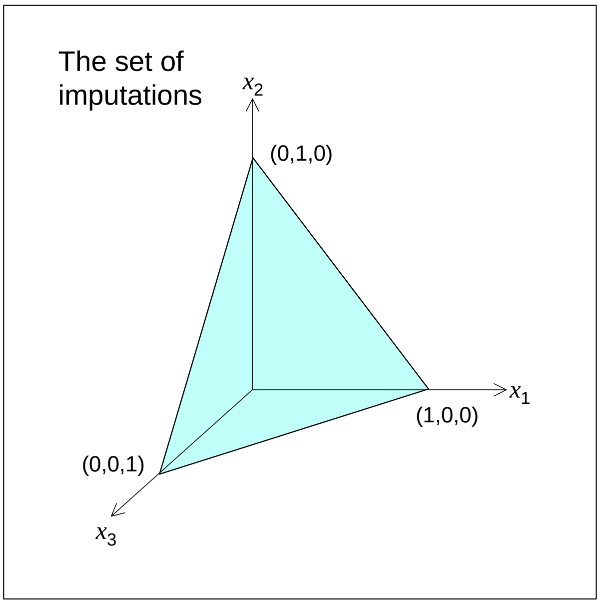
As an example consider the following constant sum game with $N = 3$:

$$\begin{aligned}
&v(123) = 1 \\
&v(12) = v(13) = v(23) = 1 \\
&v(1) = v(2) = v(3) = 0
\end{aligned}$$

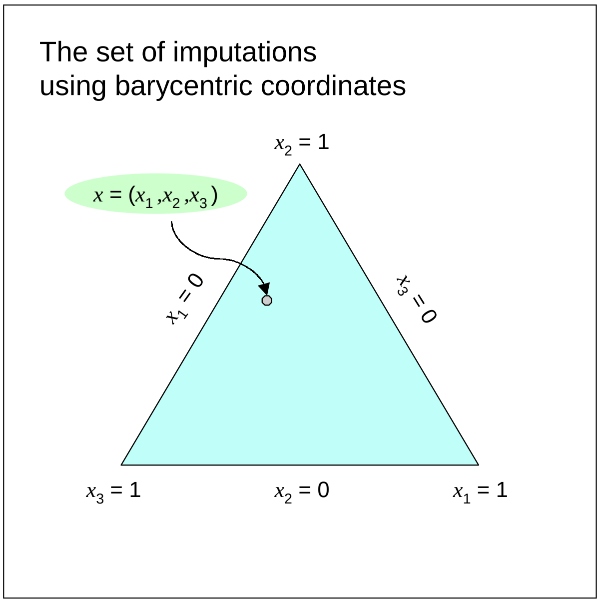
The set of imputations is

$$A(v) = \{x = (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2, 3\}$$

This set can be illustrated as a subset in $\mathbb{R}^3$ as follows:

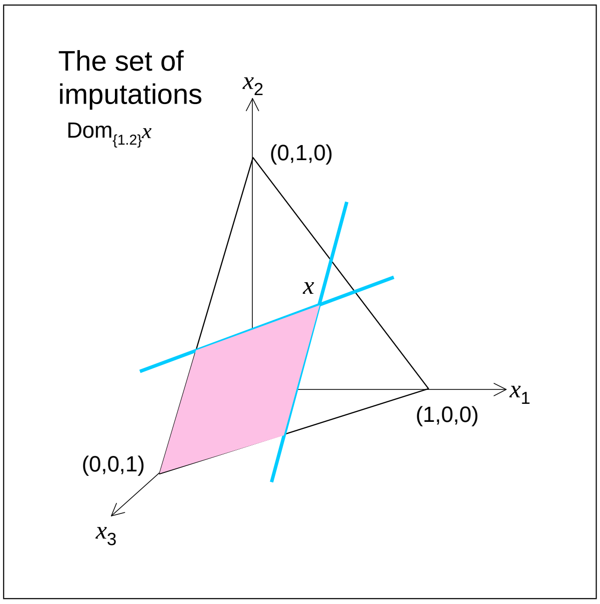


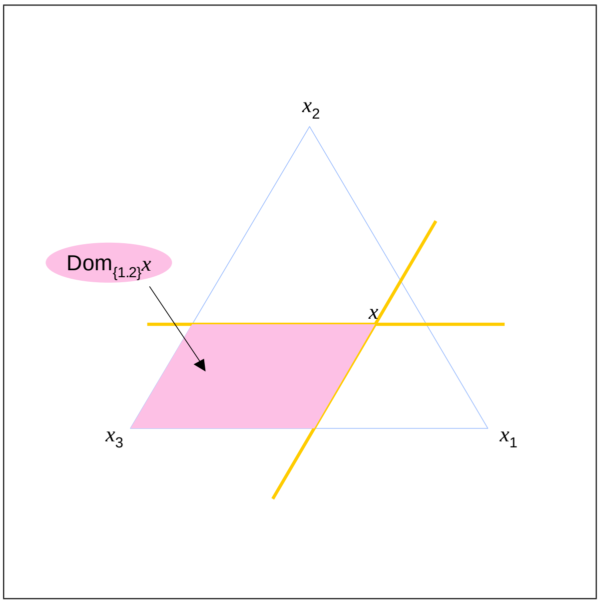
or alternatively, using barycentric coordinates



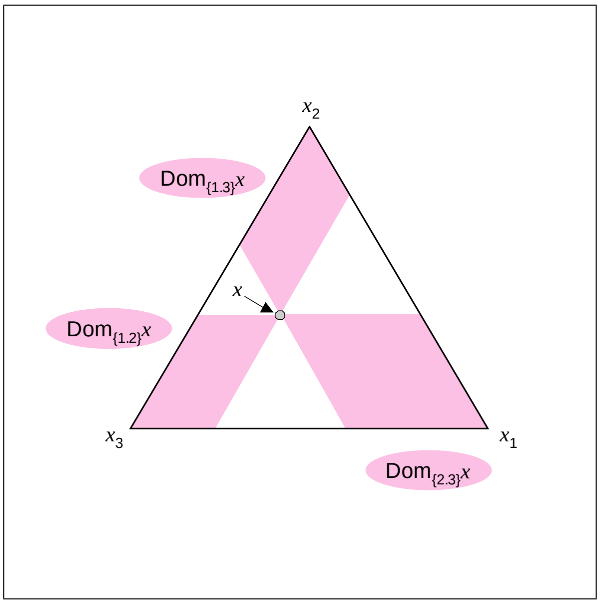
For an interior point x we get

$$\text{Dom}_{\{1,2\}} x = A(v) \cap \{y \,|\, y_1 < x_1 \text{ and } y_2 < x_2\}$$





And for all two-player coalitions we obtain



Question 5.1. Prove that

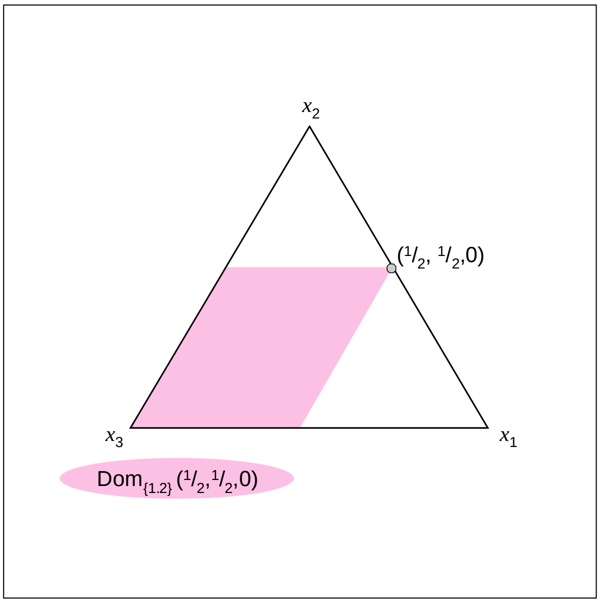
(1)	$\text{Dom}_N A(v)$	$=$	$\emptyset$	
(2)	$\text{Dom}_{\{i\}} A(v)$	$=$	$\emptyset$	$\forall i$
(3)	$\mathcal{C}$	$=$	$\emptyset$	

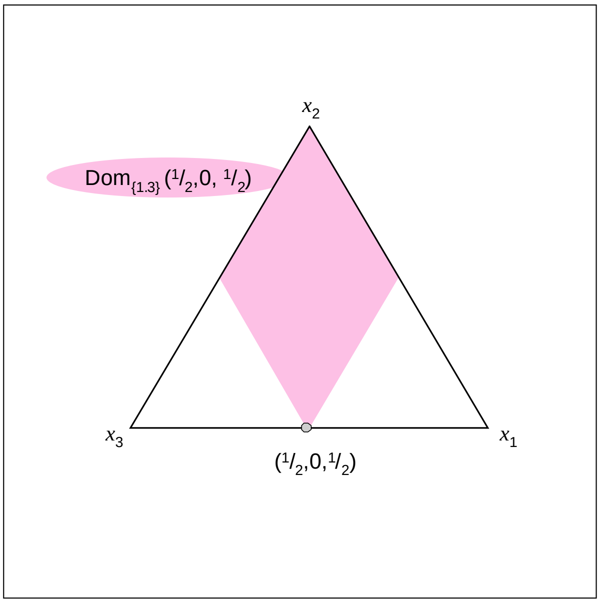
Note that (1) and (2) are general statements, while (3) is true for this particular game.

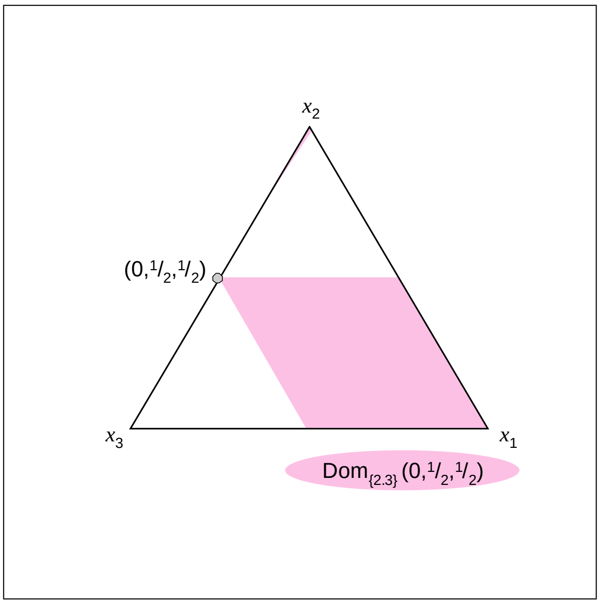
Now consider the set

$$K = \{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{2}, 0, \tfrac{1}{2}), (0, \tfrac{1}{2}, \tfrac{1}{2})\}$$

and note that the sets $\mathsf{Dom}_{\{1,2\}}\,(\frac{1}{2},\frac{1}{2},0)$, $\mathsf{Dom}_{\{1,3\}}\,(\frac{1}{2},0,\frac{1}{2})$, and $\mathsf{Dom}_{\{2,3\}}\,(0,\frac{1}{2},\frac{1}{2})$ can be illustrated as follows:







We will let you verify that

1. $K \cap \text{Dom}\, K = \varnothing$
2. $K \cup \text{Dom}\, K = A(v)$

so that K is a stable set.

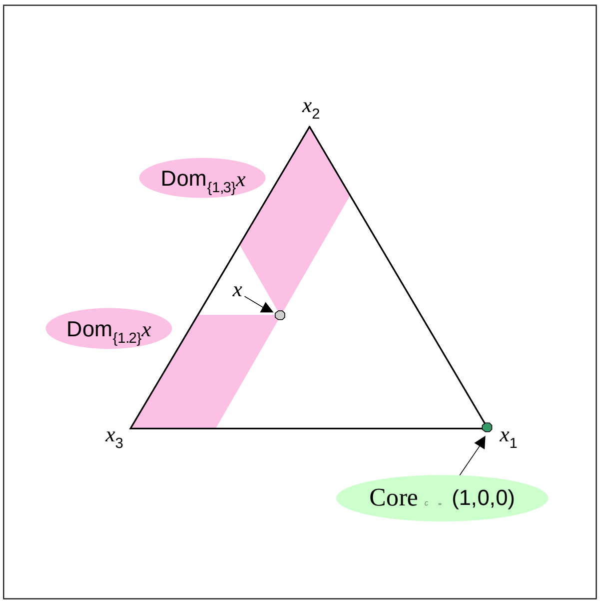
Question 5.2. There are more stable sets (an uncountable collection). Find them, and show that, for this example,

$$\begin{aligned} \cap_\alpha K_\alpha &= \varnothing \\ \cup_\alpha K_\alpha &= A(v) \end{aligned}$$

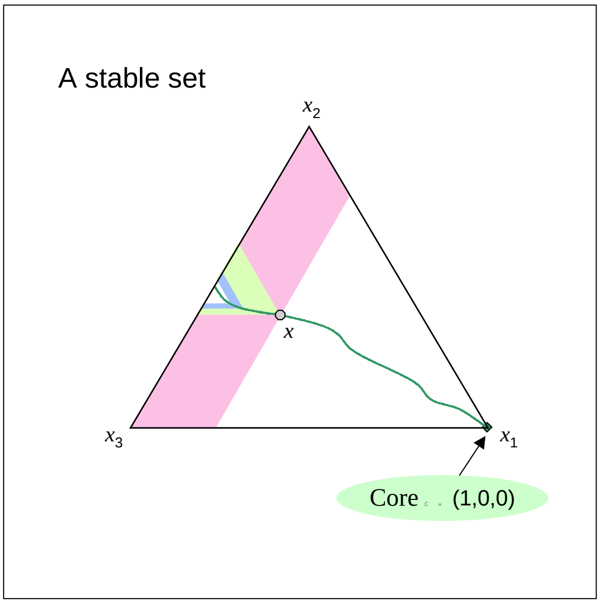
Now, let's look at the veto game:

$$\begin{aligned} &v(123) = 1 \\ &v(12) = v(13) = 1 \\ &v(23) = v(1) = v(2) = v(3) = 0 \end{aligned}$$

This game has a core at $(1, 0, 0)$ as shown in the following diagram:



Question 5.3. Verify that any continuous curve from $\mathcal{C}$ to the surface $x_2 + x_3 = 1$ with a Lipshitz condition of $30°$ or less is a stable set.



Note that

$$\begin{aligned} \cap_\alpha K_\alpha &= \mathcal{C} \\ \cup_\alpha K_\alpha &= A(v) \end{aligned}$$

5.3 Nomenclature

Much of this section is from Willick [13].

5.3.1 Coalition structures

A *coalition structure* is any partition of the player set into coalitions Let $N = \{1, 2, \ldots, n\}$ denote the set of n players.

Definition 5.2. *A* **coalition structure***, $\mathcal{P}$, is a partition of N into non-empty sets such that $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$ where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^M R_i = N$.*

5.3.2 Partition function form

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \ldots, \{n\}\}$ denote the singleton coalition structure. The coalition containing all players N is called the *grand coalition*. The coalition structure $\mathcal{P}_N \equiv \{N\}$ is called the *grand coalition structure*.

In *partition function form games*, the value of a coalition, S, can depend on the coalition arrangement of players in $N - S$ (See Lucas and Thrall [11]).

Definition 5.3. *The game (N, v) is a n-***person game in partition function form** *if $v(S, \mathcal{P})$ is a real valued function which assigns a number to each coalition $S \in \mathcal{P}$ for every coalition structure $\mathcal{P}$.*

5.3.3 Superadditivity

A game is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

Most non-superadditive games can be mapped into superadditive games. The following reason is often given: Suppose there exist disjoint coalitions S and T such that

$$v(S \cup T) < v(S) + v(T)$$

Then S and T could secretly form the coalition $S \cup T$ and collect the value $v(S) + v(T)$. The coalition $S \cup T$ would then divide the amount among its total membership.

Definition 5.4. *The game v is said to be the* **superadditive cover** *of the game u if for all $P \subseteq N$,*

$$v(P) = \max_{\mathcal{P}_P^*} \sum_{R \in \mathcal{P}_P^*} u(R)$$

where $\mathcal{P}_P^*$ be a partition of P.

Note 5.4. $\mathcal{P}_P^*$ is a coalition structure restricted to members of P

Note 5.5. A problem with using a superadditive cover is that it requires the ingredient of secrecy. Yet all of the players are assumed to have perfect information.

It also requires a dynamic implementation process. The players need to first decide on their secret alliance, then collect the payoffs as S and T individually, and finally divide the proceeds as $S \cup T$. But characteristic function form games are assumed to be static.

Example 5.3. Consider this three-person game:

$$\begin{aligned}
&u(123) = 1 \\
&u(12) = u(13) = u(23) = 1 \\
&u(2) = u(3) = 0 \\
&u(1) = 5
\end{aligned}$$

Note that (N, u) is not superadditive. The superadditive cover of (N, u) is

$$\begin{aligned}
&v(123) = 6 \\
&v(12) = 5 \\
&v(13) = 5 \\
&v(23) = 1 \\
&v(2) = v(3) = 0 \\
&v(1) = 5
\end{aligned}$$

We can often relax the requirement of superadditivty and assume only that the grand coalition obtains a value at least as great as the sum of the values of any partition of the grand coalition. Such games are called *cohesive*.

Definition 5.5. *A characteristic function game is said to be* **cohesive** *if*

$$v(N) = \max_{\mathcal{P}} \sum_{P \in \mathcal{P}} v(P).$$

There are important examples of cohesive games. For instance, we will see later that some models of hierarchical organizations produce cohesive games that are not superadditive.

5.3.4 Essential games

Definition 5.6. *A game is* **essential** *if*

$$\sum_{i \in N} v(\{i\}) < v(N)$$

A game is **inessential** *if*

$$\sum_{i \in N} v(\{i\}) \geq v(N)$$

Note 5.6. If $\sum_{i \in N} v(i) > v(N)$ then $A(v) = \emptyset$. If $\sum_{i \in N} v(i) = v(N)$ then $A(v) = \{(v(1), v(2), \ldots, v(n))\}$

5.3.5 Constant sum games

Definition 5.7. *A game is a* **constant sum game** *if*

$$v(S) + v(N - S) = v(N) \quad \forall\, S \subset N$$

5.3.6 Strategic equivalence

Definition 5.8. *Two games (N, v_1) and (N, v_2) are* **strategically equivalent** *if and only if there exist $c > 0$ and scalars $a_1, \ldots, a_n$ such that*

$$v_1(S) = cv_2(S) + \sum_{i \in S} a_i \quad \forall\, M \subseteq N$$

Properties of strategic equivalence:

1. It's a linear transformation
2. It's an equivalence relation
   - reflexive
   - symmetric
   - transitive

   Hence it partitions the set of games into equivalence classes.
3. It's an isomorphism with respect to dom on $A(v_2) \to A(v_1)$. So, strategic equivalence preserves important solution concepts.

5.3.7 Normalization

Definition 5.9. *A game (N, v) is in $(0, 1)$* **normal form** *if*

$$\begin{aligned} v(N) &= 1 \\ v(\{i\}) &= 0 \quad \forall\, i \in N \end{aligned}$$

The set $A(v)$ for a game in $(0, 1)$ normal form is a "probability simplex."

Suppose a game is in $(0, 1)$ normal form and superadditive, then $0 \leq v(S) \leq 1$ for all $S \subseteq N$.

An essential game (N, u) can be converted to $(0, 1)$ normal form by using

$$v(S) = \frac{u(S) - \sum_{i \in S} u(\{i\})}{u(N) - \sum_{i \in N} u(\{i\})}$$

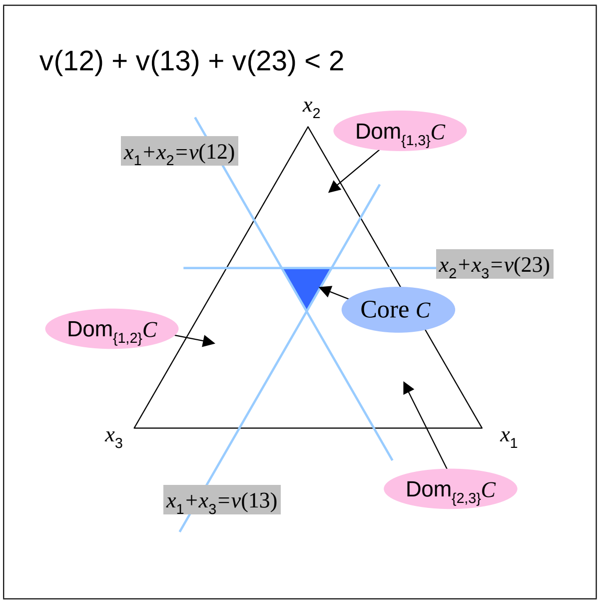
Note that the denominator must be positive for any essential game (N, u).

Note 5.7. For $N = 3$ a game in $(0, 1)$ normal form can be completely defined by specifying $(v(12), v(13), v(23))$.

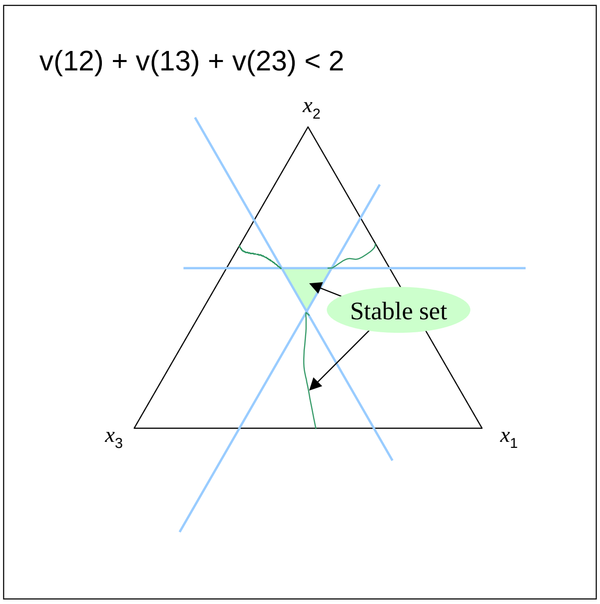
Question 5.4. Show that $\mathcal{C} \neq \emptyset$ for any three-person $(0, 1)$ normal form game with

$$v(12) + v(13) + v(23) < 2$$

Here's an example:[2]



Show that stable sets are of the following form:



[2]My thanks to Ling Wang for her suggestions on this section.

Produce similar diagrams for the case $v(12) + v(13) + v(23) > 2$.

Is $\mathcal{C} = \emptyset$ for $v(12) + v(13) + v(23) = 2$?

5.4 Garbage game

There are N players. Each player produces one bag of garbage and dumps it in another's yard. The payoff for any player is

$$-1 \times (\text{the number of bags in his yard})$$

We get

$$\begin{aligned} v(N) &= -n \\ v(M) &= |M| - n \quad \text{for } |M| < n \end{aligned}$$

We have $\mathcal{C} = \emptyset$ when $n > 2$. To show this, note that $x \in \mathcal{C}$ implies

$$\sum_{i \in N - \{j\}} x_i \geq v(N - \{j\}) = -1 \quad \forall\, j \in N$$

Summing over all $j \in N$,

$$\begin{aligned} (n-1) \sum_{i \in N} x_i &\geq -n \\ (n-1) v(N) &\geq -n \\ (n-1)(-n) &\geq -n \\ n &\leq 2 \end{aligned}$$

5.5 Pollution game

There are n factories around a lake.

Input water is free, but if the lake is dirty, a factory may need to pay to clean the water. If k factories pollute the lake, the cost to a factory to clean the incoming water is kc.

Output water is dirty, but a factory might pay to treat the effluent at a cost of b.

Assume $0 < c < b < nc$.

If a coalition M forms, all of it's members could agree to pollute with a payoff of $|M|(-nc)$. Or, all of it's members could agree to clean the water with a payoff of $|M|(-(n-|M|)c) - |M|b$. Hence,

$$\begin{aligned} v(M) &= \max\{\{|M|(-nc)\}, \{|M|(-(n-|M|)c) - |M|b\}\} \quad \text{for } M \subset N \\ v(N) &= \max\left\{\{-n^2 c\}, \{-nb\}\right\} \end{aligned}$$

Question 5.5. Show that $\mathcal{C} \neq \emptyset$ and $x = (-b, \ldots, -b) \in \mathcal{C}$.

5.6 Balanced sets and the core

The presentation in this section is based on Owen [9]

The core $\mathcal{C}$ can be defined as the set of all $(x_1, \ldots, x_n) \in A(V) \subset \mathbb{R}^n$ such that

$$\begin{aligned} \sum_{i \in N} x_i \equiv x(N) &= v(N) \quad \text{and} \\ \sum_{i \in S} x_i \equiv x(S) &\geq v(S) \quad \forall\, S \in 2^N \end{aligned}$$

If we further define an **additive set function** $x(\cdot)$ as any function such that

$$\begin{aligned} &x : 2^N \to \mathbb{R} \\ &x(S) = \sum_{i \in S} x(\{i\}) \end{aligned}$$

we get the following, equivalent, definition of a core:

Definition 5.10. *The core $\mathcal{C}$ of a game (N, v) is the set of additive $x : 2^N \to \mathbb{R}$ such that*

$$\begin{aligned} x(N) &= v(N) \\ x(S) &\geq v(S) \quad \forall S \subset N \end{aligned}$$

We would like to characterize those characteristic functions v for which the core is nonempty.

Note that $\mathcal{C} \neq \emptyset$ if and only if the linear programming problem

$$\begin{aligned} \min \quad & z = \textstyle\sum_{i=1}^n x_i \\ \text{st:} \quad & \textstyle\sum_{i \in S} x_i \geq v(S) \quad \forall\, S \subset N \end{aligned} \tag{4}$$

has a minimum $z^* \le v(N)$.

Consider the dual to the above linear programming problem (4). That is,

$$\begin{array}{rlcll} & \max \ \sum_{S \subset N} y_S v(S) & = & q & \\ (5) \qquad & \text{st:} \ \sum_{S \ni i} y_S & = & 1 & \forall\, i \in N \\ & y_S & \ge & 0 & \forall\, S \subset N \end{array}$$

Note that both the linear program (4) and its dual (5) are always feasible. So

$$\min z = \max q$$

by the duality theorem. Hence, the core is nonempty if and only if

$$\max q \le v(N)$$

This leads to the following:

Theorem 5.2. *A necessary and sufficient condition for the game (N, v) to have $\mathcal{C} \ne \emptyset$ is that for every nonnegative vector $(y_S)_{S \subset N}$ satisfying*

$$\sum_{S \ni i} y_S = 1 \quad \forall\, i$$

we have

$$\sum_{S \subset N} y_S v(S) \le v(N)$$

To make this more useful, we introduce the concept of a *balanced collection* of coalitions.

Definition 5.11. $\mathcal{B} \subset 2^N$ *is* **balanced** *if there exists $y_S \in \mathbb{R}$ with $y_S > 0$ for all $S \in \mathcal{B}$ such that*

$$\sum_{S \ni i} y_S = 1 \quad \forall\, i \in N$$

y is called the **balancing vector** (or weight vector) for $\mathcal{B}$. The individual y_S's are called **balancing coefficients**.

Example 5.4. Suppose $N = \{1, 2, 3\}$

$\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$ is a balanced collection with $y_{\{1\}} = 1$, $y_{\{2\}} = 1$, and $y_{\{3\}} = 1$.

$\mathcal{B} = \{\{1, 2\}, \{3\}\}$ is a balanced collection with $y_{\{1,2\}} = 1$ and $y_{\{3\}} = 1$.

$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is a balanced collection with $y_{\{1,2\}} = \frac{1}{2}$, $y_{\{1,3\}} = \frac{1}{2}$, and $y_{\{2,3\}} = \frac{1}{2}$.

Theorem 5.3. *The union of balanced collections is balanced.*

Lemma 5.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be balanced collections such that $\mathcal{B}_1 \subset \mathcal{B}_2$ but $\mathcal{B}_1 \neq \mathcal{B}_2$. Then there exists a balanced collection $\mathcal{B}_3 \neq \mathcal{B}_2$ such that $\mathcal{B}_3 \cup \mathcal{B}_1 = \mathcal{B}_2$.*

The above lemma leads us to define the following:

Definition 5.12. *A* **minimal balanced collection** *is a balanced collection for which no proper subcollection is balanced.*

Theorem 5.4. *Any balanced collection can be written as the union of minimal balanced collections.*

Theorem 5.5. *Any balanced collection has a unique balancing vector if and only if it is a minimal balanced collection.*

Theorem 5.6. *Each extreme point of the polyhedron for the dual linear programming problem (5) is the balancing vector of a minimal balanced collection.*

Corollary 5.1. *A minimal balanced collection has at most n sets.*

The result is the following theorem:

Theorem 5.7. (Shapley-Bondareva) *The core is nonempty if and only if for every minimal balanced collection $\mathcal{B}$ with balancing coefficients $(y_S)_{S\in\mathcal{B}}$ we have*

$$v(N) \geq \sum_{s\in\mathcal{B}} y_S v(S)$$

Example 5.5. Let $N = \{1, 2, 3\}$. Besides the partitions, such as $\{\{1, 2\}, \{3\}\}$, there is only one other minimal balanced collection, namely,

$$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

with

$$y = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

Therefore a three-person game (N, v) has a nonempty core if and only if

$$\begin{aligned} \tfrac{1}{2}v(\{1,2\}) + \tfrac{1}{2}v(\{1,3\}) + \tfrac{1}{2}v(\{2,3\}) &\leq v(N) \\ v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) &\leq 2v(N) \end{aligned}$$

Question 5.6. Use the above result and reconsider Question 5.4 on page 5-14.

Question 5.7. Suppose we are given $v(S)$ for all $S \neq N$. What is the smallest value of $v(N)$ such that $\mathcal{C} \neq \emptyset$?

5.7 The Shapley value

Much of this section is from Yang [14].

Definition 5.13. *A* **carrier** *for a game (N, v) is a coalition $T \subseteq N$ such that $v(S) \leq v(S \cap T)$ for any $S \subseteq N$.*

The above definition is slightly different from the one given by Shapley [10]. Shapley uses $v(S) = v(S \cap T)$ instead of $v(S) \leq v(S \cap T)$. However, when the game (N, v) is superadditive, Shapley's definition and Yang's definition are equivalent.

A carrier is a group of players with the ability to benefit the coalitions they join. A coalition can remove any of its members who do not belong to the carrier and get the same, or greater value.

Let $\Pi(N)$ denote the set of all permutations on N, that is, the set of all one-to-one mappings from N onto itself.

Definition 5.14. (Owen [9]) *Let (N, v) be an n-person game, and let $\pi \in \Pi(N)$. Then, the game $(N, \pi v)$ is defined as the game (N, u), such that*

$$u(\{\pi(i_1), \pi(i_2), \ldots, \pi(i_{|S|})\}) = v(S)$$

for any coalition $S = \{i_1, i_2, \ldots, i_{|S|}\}$.

Definition 5.15. (Friedman [3]) *Let (N, v) be an n-person game. The* **marginal value,** *$c_S(v)$, for coalition $S \subseteq N$ is given by*

$$c_{\{i\}}(v) \equiv v(\{i\})$$

for all $i \in N$, and

$$c_S(v) \equiv v(S) - \sum_{L \subset S} c_L(v)$$

for all $S \subseteq N$ *with* $|S| \geq 2$.

The marginal value of S can also be computed by using the formula

$$c_S(v) = \sum_{L \subset S} (-1)^{|S|-1} v(L).$$

5.7.1 The Shapley axioms

Let $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ be an n-dimensional vector satisfying the following three axioms:

Axiom S 1. (Symmetry) *For each* $\pi \in \Pi(N)$, $\phi_{\pi(i)}(\pi v) = \phi_i(v)$.

Axiom S 2. (Efficiency) *For each carrier* C *of* (N, v)

$$\sum_{i \in C} \phi_i(v) = v(C).$$

Axiom S 3. (Law of Aggregation) *For any two games* (N, v) *and* (N, w)

$$\phi(v + w) = \phi(v) + \phi(w).$$

Theorem 5.8. (Shapley [10]) *For any superadditive game* (N, v) *there is a unique vector of values* $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$ *satisfying the above three axioms. Moreover, for each player* i *this value is given by*

$$\phi_i(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{|S|} c_S(v) \tag{6}$$

Note 5.8. The Shapley value can be equivalently written [9] as

$$\phi_i(v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \left(\frac{(|T|-1)!(n-|T|)!}{n!} \right) [v(T) - v(T - \{i\})] \tag{7}$$

This formula can be interpreted as follows: Suppose n players arrive one after the other into a room that will eventually contain the grand coalition. Consider all possible sequencing arrangements of the n players. Suppose that any sequence can occur with probability $\frac{1}{n!}$. If Player i arrives and finds coalition $T - \{i\}$ already in the room, his contribution to the coalition is $v(T) - v(T - \{i\})$. The Shapley value is the expected value of the contribution of Player i.

5.8 A generalization of the Shapley value

Suppose we introduce the concept of taxation (or resource redistribution) and relax just one of the axioms. Yang [14], has shown that the Shapley value and the **egalitarian value**

$$\phi_i^0(v) = \frac{v(N)}{n} \quad \forall\, i \in N$$

are then the extremes of an entire family of values for all cohesive (not necessarily superadditive) games.

Axiom Y 1. (Symmetry) *For each* $\pi \in \Pi(N)$, $\psi_{\pi(i)}(\pi v) = \psi_i(v)$.

Axiom Y 2. (Rationing) *For each carrier* C *of* (N, v)

$$\sum_{i \in C} \psi_i(v) = g(C)v(C) \quad \textit{with } \tfrac{|C|}{n} \leq g(C) \leq 1.$$

Axiom Y 3. (Law of Aggregation) *For any two games* (N, v) *and* (N, w)

$$\psi(v + w) = \psi(v) + \psi(w).$$

Note that Yang only modifies the second axiom. The function $g(C)$ is called the *rationing function*. It can be any real-valued function defined on attributes of the carrier C with range $\left[\frac{|C|}{n}, 1\right]$. If the game (N, v) is superadditive, then $g(C) = 1$ yields Shapley's original axioms.

A particular choice of the rationing function $g(C)$ produces a convex combination between the egalitarian value and the Shapley value. Let $N = \{1, \ldots, n\}$ and let $c \equiv |C|$ for $C \subseteq N$. Given the value of the parameter $r \in [\frac{1}{n}, 1]$ consider the real-valued function

$$g(C) \equiv g(c, r) = \frac{(n - c)r + (c - 1)}{n - 1}.$$

The function $g(C)$ specifies the distribution of revenue among the players of a game.

Note that this function can be rewritten as

$$g(c, r) = 1 - (1 - r)\left(\frac{n - c}{n - 1}\right).$$

For games with a large number of players,

$$\lim_{n\to\infty} g(c,r) = r \in (0,1]$$

so that $(1-r)$ can be regarded as a "tax rate" on carriers.

This results in the following theorem:[3]

Theorem 5.9. *Let (N,v) be a cohesive n-person cooperative transferable utility game. For each $r \in [\frac{1}{n},1]$, there exists a unique value, $\psi_{i,r}(v)$, for each Player i satisfying the three axioms with rationing function*

$$g(C) = \frac{(n-c)r + (c-1)}{n-1}.$$

Moreover, this unique value is given by

$$\psi_{i,r}(v) = (1-p)\phi_i(v) + p\frac{v(N)}{n} \quad \forall\, i \in N \tag{8}$$

where $p = \dfrac{n-nr}{n-1} \in (0,1)$.

Note that the rationing function can be written[4] in terms of $p \in (0,1)$ as

$$g(c,p) = p + (1-p)\frac{c}{n}$$

Example 5.6. Consider a two-person game with

$$v(\{1\}) = 1, \quad v(\{2\}) = 0, \quad v(\{1,2\}) = 2$$

Player 2 can contribute 1 to a coalition with Player 1. But, Player 1 can get 1 on his own, leaving Player 2 with nothing.

The family of values is

$$\psi_r(v) = \left(\frac{1}{2}+r, \frac{3}{2}-r\right)$$

for $\frac{1}{2} \le r \le 1$. The Shapley value (with $r=1$) is $\left(\frac{3}{2},\frac{1}{2}\right)$.

Example 5.7. Consider a modification of the above game in Example (5.6) with

$$v(\{1\}) = 1, \quad v(\{2\}) = 0, \quad v(\{1,2\}) = 1$$

[3] We are indebted to an anonymous reviewer for the simplified version of this theorem.

[4] Once again, our thanks to the same anonymous reviewer for this observation.

In this case, Player 2 is a dummy player.

The family of values is

$$\psi_r(v) = (r, 1 - r)$$

for $\frac{1}{2} \leq r \leq 1$. The Shapley value (with $r = 1$) is $(1, 0)$.

Example 5.8. This solution approach can be applied to a problem suggested by Nowak and Radzik [8]. Consider a three-person game where

$$\begin{gathered} v(\{1\}) = v(\{2\}) = 0, \quad v(\{3\}) = 1, \\ v(\{1,2\}) = 3.5, \quad v(\{1,3\}) = v(\{2,3\}) = 0, \\ v(\{1,2,3\}) = 5. \end{gathered}$$

The Shapley value for this game is

$$\phi(v) = \left(\tfrac{25}{12}, \tfrac{25}{12}, \tfrac{10}{12}\right).$$

Note that the Shapley value will not necessarily satisfy the condition of *individual rationality*

$$\phi_i(v) \geq v(\{i\})$$

when the characteristic function v is not superadditive. That is the case here since $\phi_3(v) < v(\{3\})$.

The *solidarity value* (Nowak and Radzik [8]) $\xi(v)$ of this game is

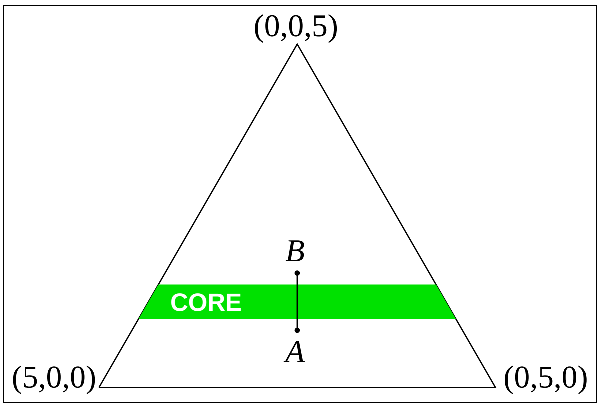
$$\xi(v) = \left(\tfrac{16}{9}, \tfrac{16}{9}, \tfrac{13}{9}\right)$$

and is in the core of (N, v).

For every $r \in [\frac{1}{n}, 1]$, the general form of the family of values is

$$\psi_r(v) = \left(\frac{35 + 15r}{24}, \frac{35 + 15r}{24}, \frac{50 - 30r}{24}\right).$$

The diagram in the following figure shows the relationship between the family of values and the core.



Note that, in the diagram,

$$\begin{aligned} A &= \left(\tfrac{25}{12}, \tfrac{25}{12}, \tfrac{10}{12}\right) \quad \text{(the Shapley value)} \\ B &= \left(\tfrac{5}{3}, \tfrac{5}{3}, \tfrac{5}{3}\right). \end{aligned}$$

Neither of these extreme values of the family of values is in the core for this game. However, those solutions for $\frac{7}{15} \leq r \leq \frac{13}{15}$ are elements of the core.

Example 5.9. Nowak and Radzik [8] offer the following example related to social welfare and income redistribution: Players 1, 2, and 3 are brothers living together. Players 1 and 2 can make a profit of one unit, that is, $v(\{1, 2\}) = 1$. Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, $v(\{1, 2, 3\}) = 1$. Also, $v(\{1, 3\}) = v(\{2, 3\}) = 0$ and $v(\{i\}) = 0$ for every Player i.

Shapley value of this game is

$$\phi(v) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

and for the family of values, we get

$$\psi_r(v) = \left(\frac{1+r}{4}, \frac{1+r}{4}, \frac{1-r}{2}\right)$$

for $r \in [\frac{1}{3}, 1]$. Every r yields a solution satisfying individual rationality, but, in this case, $\psi_r(v)$ belongs to the core only when it equals the Shapley value ($r = 1$).

For this particular game, the solidarity value is a member of the family when $r = \frac{5}{9}$. Nowak and Radzik propose this single value as a "better" solution for the game (N, v) than its Shapley value. They suggest that it could be used to include subjective social or psychological aspects in a cooperative game.

Question 5.8. Suppose game (N, v) has core $\mathcal{C} \neq \text{Ø}$. Let

$$\mathcal{F} \equiv \{\psi_r(v) \mid \frac{1}{n} \leq r \leq 1\}$$

denote the set of Yang's values when using rationing function $g(c, r)$. Under what conditions will $\mathcal{C} \cap \mathcal{F} \neq \text{Ø}$?

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