IE675 Game Theory

Lecture Note Set 3

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3 N-PERSON GAMES

3.1 N-Person Games in Strategic Form

3.1.1 Basic ideas

We can extend many of the results of the previous chapter for games with $N > 2$ players.

Let $M_i = \{1, \ldots, m_i\}$ denote the set of m_i pure strategies available to Player i.

Let $n_i \in M_i$ be the strategy actually selected by Player i, and let $a^i_{n_1,n_2,\ldots,n_N}$ be the payoff to Player i if

$$\begin{array}{l} \text{Player 1 chooses strategy } n_1 \\ \text{Player 2 chooses strategy } n_2 \\ \quad\vdots \\ \text{Player } N \text{ chooses strategy } n_N \end{array}$$

Definition 3.1. *The strategies* $(n_1^*, \ldots, n_N^*)$ *with* $n_i^* \in M_i$ *for all* $i \in N$ *form a Nash equilibrium solution if*

$$a^1_{n_1^*,n_2^*,\ldots,n_N^*} \;\geq\; a^1_{n_1,n_2^*,\ldots,n_N^*} \quad \forall\, n_i \in M_1$$

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$$
\begin{aligned}
a^2_{n_1^*,n_2^*,\ldots,n_N^*} &\geq a^2_{n_1^*,n_2,\ldots,n_N^*} \quad \forall\, n_2 \in M_2 \\
&\vdots \\
a^N_{n_1^*,n_2^*,\ldots,n_N^*} &\geq a^N_{n_1^*,n_2^*,\ldots,n_N} \quad \forall\, n_N \in M_N
\end{aligned}
$$

Definition 3.2. *Two N-person games with payoff functions $a^i_{n_1,n_2,\ldots,n_N}$ and $b^i_{n_1,n_2,\ldots,n_N}$ are* **strategically equivalent** *if there exists $\alpha_i > 0$ and scalars β_i for $i = 1, \ldots, n$ such that*

$$
a^i_{n_1,n_2,\ldots,n_N} = \alpha_i b^i_{n_1,n_2,\ldots,n_N} + \beta_i \quad \forall\, i \in N
$$

3.1.2 Nash solutions with mixed strategies

Definition 3.3. *The mixed strategies $(y^{*1}, \ldots, y^{*N})$ with $y^{*i} \in \Xi^{M_i}$ for all $i \in N$ form a Nash equilibrium solution if*

$$
\begin{aligned}
\sum_{n_1} \cdots \sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^1_{n_1,\ldots,n_N} &\geq \sum_{n_1} \cdots \sum_{n_N} y^{1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^1_{n_1,\ldots,n_N} \quad \forall\, y^1 \in \Xi^{M_1} \\
\sum_{n_1} \cdots \sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^2_{n_1,\ldots,n_N} &\geq \sum_{n_1} \cdots \sum_{n_N} y^{*1}_{n_1} y^{2}_{n_2} \cdots y^{*N}_{n_N} a^2_{n_1,\ldots,n_N} \quad \forall\, y^2 \in \Xi^{M_2} \\
&\vdots \\
\sum_{n_1} \cdots \sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^N_{n_1,\ldots,n_N} &\geq \sum_{n_1} \cdots \sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{N}_{n_N} a^N_{n_1,\ldots,n_N} \quad \forall\, y^N \in \Xi^{M_N}
\end{aligned}
$$

Note 3.1. Consider the function

$$
\begin{aligned}
\psi^i_{n_i}(y^1, \ldots, y^n) = &\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \\
&- \sum_{n_1} \cdots \sum_{n_{i-1}} \sum_{n_{i+1}} \cdots \sum_{n_N} y^1_{n_1} \cdots y^{i-1}_{n_{i-1}} y^{i+1}_{n_{i+1}} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}
\end{aligned}
$$

This represents the difference between the following two quantities:

1. the expected payoff to Player i if all players adopt mixed strategies $(y^1, \ldots, y^N)$:

$$
\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}
$$

2. the expected payoff to Player i if all players except Player i adopt mixed strategies $(y^1, \ldots, y^N)$ and Player i uses pure strategy n_i:

$$\sum_{n_1} \cdots \sum_{n_{i-1}} \sum_{n_{i+1}} \cdots \sum_{n_N} y^1_{n_1} \cdots y^{i-1}_{n_{i-1}} y^{i+1}_{n_{i+1}} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}$$

Remember that the mixed strategies include the pure strategies. For example, $(0, 1, 0, \ldots, 0)$ is a mixed strategy that implements pure strategy 2.

For example, in a two-player game, for each $n_1 \in M_1$ we have

$$\begin{aligned} \psi^1_{n_1}(y^1, y^2) &= \left[y^1_1 y^2_1 a^1_{11} + y^1_1 y^2_2 a^1_{12} + y^1_2 y^2_1 a^1_{21} + y^1_2 y^2_2 a^1_{22}\right] \\ &\quad - \left[y^2_1 a^1_{n_1 1} + y^2_2 a^1_{n_1 2}\right] \end{aligned}$$

The first term

$$y^1_1 y^2_1 a^1_{11} + y^1_1 y^2_2 a^1_{12} + y^1_2 y^2_1 a^1_{21} + y^1_2 y^2_2 a^1_{22}$$

is the expected value if Player 1 uses mixed strategy y_1. The second term

$$y^2_1 a^1_{n_1 1} + y^2_2 a^1_{n_1 2}$$

is the expected value if Player 1 uses pure strategy n_1. Player 2 uses mixed strategy y_2 in both cases.

The next theorem (Theorem 3.1) will guarantee that every game has at least one Nash equilibrium in mixed strategies. Its proof depends on things that can go wrong when $\psi^i_{n_i}(y^1, \ldots, y^n) < 0$. So we will define

$$c^i_{n_i}(y^1, \ldots, y^n) = \min\{\psi^i_{n_i}(y^1, \ldots, y^n), 0\}$$

The proof of Theorem 3.1 then uses the expression

$$\overline{y}^i_{n_i} = \frac{y^i_{n_i} + c^i_{n_i}}{1 + \sum_{j \in M_i} c^i_j}$$

Note that the denominator is the sum (taken over n_i) of the terms in the numerator. If all of the c^i_j vanish, we get

$$\overline{y}^i_{n_i} = y^i_{n_i}.$$

Theorem 3.1. *Every N-person finite game in normal (strategic) form has a Nash equilibrium solution using mixed strategies.*

Proof: Define $\psi^i_{n_i}$ and $c^i_{n_i}$, as above. Consider the expression

$$\overline{y}^i_{n_i} = \frac{y^i_{n_i} + c^i_{n_i}}{1 + \sum_{j \in M_i} c^i_j} \tag{1}$$

We will try to find solutions $y^i_{n_i}$ to Equation 1 such that

$$\overline{y}^i_{n_i} = y^i_{n_i} \quad \forall\, n_i \in M_i \text{ and } \forall\, i = 1, \ldots, N$$

The Brouwer fixed point theorem[1] guarantees that at least one such solution exists.

We will show that every Nash equilibrium solution is a solution to Equation 1, and that every solution to Equation 1 is a Nash equilibrium solution.

> **Remark:** First we will show that every Nash equilibrium solution is a solution to Equation 1.

Assume that $(y^{*1}, \ldots, y^{*N})$ is a Nash solution. This implies that

$$\psi^i_{n_i}(y^{1*}, \ldots, y^{n*}) \geq 0$$

which implies

$$c^i_{n_i}(y^{1*}, \ldots, y^{n*}) = 0$$

and this holds for all $n_i \in M_i$ and all $i = 1, \ldots, N$. Hence, $(y^{1*}, \ldots, y^{n*})$ solves Equation 1.

> **Remark:** Now the hard part: we must show that every solution to Equation 1 is a Nash equilibrium solution. We will do this by contradiction. That is, we will assume that a mixed strategy $(y^1, \ldots, y^N)$ is a solution to Equation 1 but is not a Nash solution. This will lead us to conclude that $(y^1, \ldots, y^N)$ is not a solution to Equation 1, a contradiction.

Assume $(y^1, \ldots, y^N)$ is a solution to Equation 1 but is not a Nash solution. Then there exists a $i \in \{1, \ldots, N\}$ (say $i = 1$) with $\tilde{y}^1 \in \Xi^{M_1}$ such that

$$\begin{aligned} &\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \\ &\qquad < \sum_{n_1} \cdots \sum_{n_N} \tilde{y}^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \end{aligned}$$

[1] The Brouwer fixed point theorem states that if S is a compact and convex subset of $\mathbb{R}^n$ and if $f : S \to S$ is a continuous function onto S, then there exists at least one $x \in S$ such that $f(x) = x$.

Rewriting the right hand side,

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}$$
$$< \sum_{n_1} \tilde{y}^1_{n_1} \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \right]$$

Now the expression

$$\left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \right]$$

is a function of n_1. Suppose this quantity is maximized when $n_1 = \tilde{n}_1$. We then get,

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}$$
$$< \sum_{n_1} \tilde{y}^1_{n_1} \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{\tilde{n}_1,\ldots,n_N} \right]$$

which yields

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \tag{2}$$
$$< \sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{\tilde{n}_1,\ldots,n_N} \tag{3}$$

> **Remark:** After this point we don't really use $\tilde{y}$ again. It was just a device to obtain $\tilde{n}_1$ which will produce our contradiction. Remember, throughout the rest of the proof, the values of $(y^1, \ldots, y^N)$ claim be a fixed point for Equation 1. If $(y^1, \ldots, y^N)$ is, in fact, not Nash (as was assumed), then we have just found a player (who we are calling Player 1) who has a *pure* strategy $\tilde{n}_1$ that can beat strategy y^1 when Players $2, \ldots, N$ use mixed strategies $(y^2, \ldots, y^N)$.

Using $\tilde{n}_1$, Player 1 obtains

$$\psi^1_{\tilde{n}_1}(y^1, \ldots, y^n) < 0$$

which means that

$$c^1_{\tilde{n}_1}(y^1, \ldots, y^n) < 0$$

which implies that

$$\sum_{j \in M_1} c_j^i < 0$$

since one of the indices in M_1 is $\tilde{n}_1$ and the rest of the c_j^i cannot be positive.

> **Remark:** Now the values $(y^1, \ldots, y^N)$ are in trouble. We have determined that their claim of being "non-Nash" produces a denominator in Equation 1 that is less than 1. All we need to do is find some pure strategy (say $\hat{n}_1$) for Player 1 with $c_{\hat{n}_i}^i(y^1, \ldots, y^n) = 0$. If we can, $(y^1, \ldots, y^N)$ will fail to be a fixed-point for Equation 1, and it will be y^1 that causes the failure. Let's see what happens...

Recall expression 2:

$$\sum_{n_1} \cdots \sum_{n_N} y_{n_1}^1 y_{n_2}^2 \cdots y_{n_N}^N a_{n_1,\ldots,n_N}^i$$

rewritten as

$$\sum_{n_1} y_{n_1}^1 \left[\sum_{n_2} \cdots \sum_{n_N} y_{n_2}^2 \cdots y_{n_N}^N a_{n_1,\ldots,n_N}^i \right]$$

and consider the term

$$\left[\sum_{n_2} \cdots \sum_{n_N} y_{n_2}^2 \cdots y_{n_N}^N a_{n_1,\ldots,n_N}^i \right] \tag{4}$$

as a function of n_1 There must be some $n_1 = \hat{n}_1$ that minimizes expression 4, with

$$\sum_{n_1} \cdots \sum_{n_N} y_{n_1}^1 y_{n_2}^2 \cdots y_{n_N}^N a_{n_1,\ldots,n_N}^i \geq \left[\sum_{n_2} \cdots \sum_{n_N} y_{n_2}^2 \cdots y_{n_N}^N a_{\hat{n}_1,n_2\ldots,n_N}^i \right]$$

For that particular strategy we have

$$\psi_{\hat{n}_1}^1(y^1, \ldots, y^n) \geq 0$$

which means that

$$c_{\hat{n}_1}^1(y^1, \ldots, y^n) = 0$$

Therefore, for Player 1, we get

$$\overline{y}_{\hat{n}_1}^1 = \frac{y_{\hat{n}_1}^1 + 0}{1 + [\text{something} < 0]} > y_{\hat{n}_1}^1$$

Hence, y^1 (which claimed to be a component of the non-Nash solution $(y^1, \ldots, y^N)$) fails to solve Equation 1. A contradiction. ■

The following theorem is an extension of a result for $N = 2$ given in Chapter 2. It provides necessary conditions for any interior Nash solution for N-person games.

Theorem 3.2. *Any mixed Nash equilibrium solution $(y^{*1}, \ldots, y^{*N})$ in the interior of $\Xi^{M_1} \times \cdots \times \Xi^{M_N}$ must satisfy*

$$
\begin{aligned}
\sum_{n_2}\sum_{n_3}\cdots\sum_{n_N} y^{*2}_{n_2} y^{*3}_{n_3} \cdots y^{*N}_{n_N} (a^1_{n_1,n_2,n_3\ldots,n_N} - a^1_{1,n_2,n_3\ldots,n_N}) &= 0 \quad \forall\, n_1 \in M_1 - \{1\} \\
\sum_{n_1}\sum_{n_3}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*3}_{n_3} \cdots y^{*N}_{n_N} (a^2_{n_1,n_2,n_3\ldots,n_N} - a^2_{n_1,1,n_3\ldots,n_N}) &= 0 \quad \forall\, n_2 \in M_2 - \{1\} \\
&\vdots \\
\sum_{n_1}\sum_{n_2}\cdots\sum_{n_{N-1}} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} (a^N_{n_1,n_2,n_3\ldots,n_N} - a^N_{n_1,n_2,n_3\ldots,1}) &= 0 \quad \forall\, n_N \in M_N - \{1\}
\end{aligned}
$$

Proof: Left to the reader.

Question 3.1. Consider the 3-player game with the following values for

$$(a^1_{n_1,n_2,n_3}, a^2_{n_1,n_2,n_3}, a^3_{n_1,n_2,n_3}) :$$

For $n_3 = 1$

	$n_2 = 1$	$n_2 = 2$
$n_1 = 1$	$(1, -1, 0)$	$(0, 1, 0)$
$n_1 = 2$	$(2, 0, 0)$	$(0, 0, 1)$

For $n_3 = 2$

	$n_2 = 1$	$n_2 = 2$
$n_1 = 1$	$(1, 0, 1)$	$(0, 0, 0)$
$n_1 = 2$	$(0, 3, 0)$	$(-1, 2, 0)$

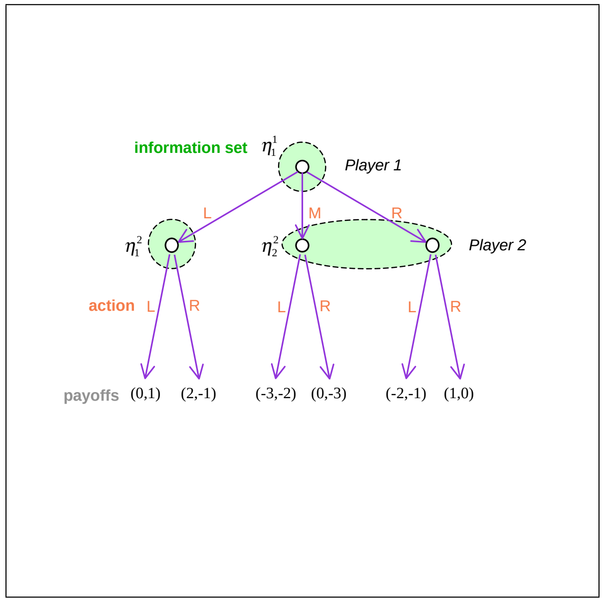
For example $a^2_{212} = 3$. Use the above method to find an interior Nash solution.

3.2 N-Person Games in Extensive Form

3.2.1 An introductory example

We will use an example to illustrate some of the issues associates with games in extensive form.

Consider a game with two players described by the following tree diagram:



Player 1 goes first and chooses an action among $\{\text{Left}, \text{Middle}, \text{Right}\}$. Player 2 then follows by choosing an action among $\{\text{Left}, \text{Right}\}$.

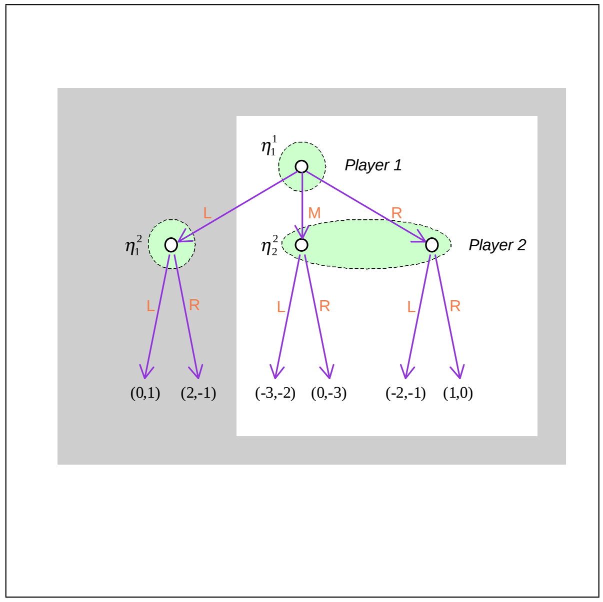
The payoff vectors for each possible combination of actions are shown at each terminating node of the tree. For example, if Player 1 chooses action $u_1 = L$ and Player 2 chooses action $u_2 = R$ then the payoff is $(2, -1)$. So, Player 1 gains 2 while Player 1 loses 1.

Player 2 does not have complete information about the progress of the game. His nodes are partitioned among two information sets $\{\eta_2^1, \eta_2^2\}$. When Player 2 chooses his action, he only knows which information set he is in, not which node.

Player 1 could analyze the game as follows:

- If Player 1 chooses $u_1 = L$ then Player 2 would respond with $u_2 = L$ resulting in a payoff of $(0, 1)$.
- If Player 1 chooses $u_1 \in \{M, R\}$ then the players are really playing the

following subgame:



which can be expressed in normal form as

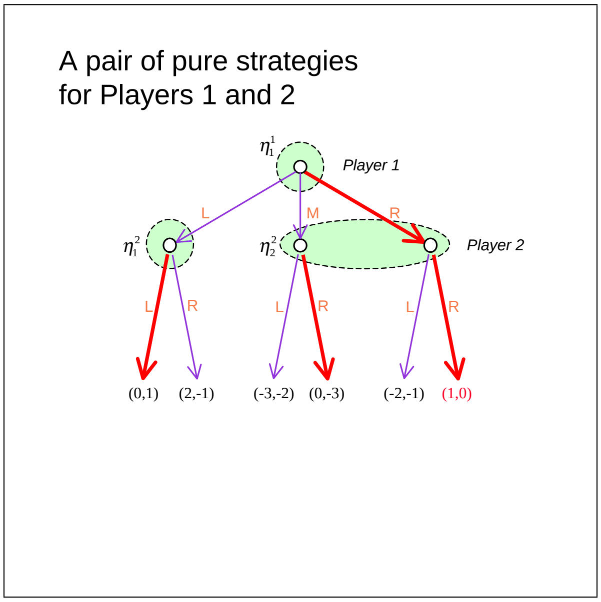
	L	R
M	(-3,-2)	(0,-3)
R	(-2,-1)	(1,0)

in which (R, R) is a Nash equilibrium strategy in pure strategies.

So it seems reasonable for the players to use the following strategies:

- For Player 1
  - If Player 1 is in information set η_1^1 choose R.
- For Player 2
  - If Player 2 is in information set η_1^2 choose L.
  - If Player 2 is in information set η_2^2 choose R.

These strategies can be displayed in our tree diagram as follows:



For games in strategic form, we denote the set of pure strategies for Player i by $M_i = \{1, \ldots, m_i\}$ and let $n_i \in M_i$ denote the strategy actually selected by Player i. We will now consider a strategy γ^i as a function whose domain is the set of information sets of Player i and whose range is the collection of possible actions for Player i. For the strategy shown above

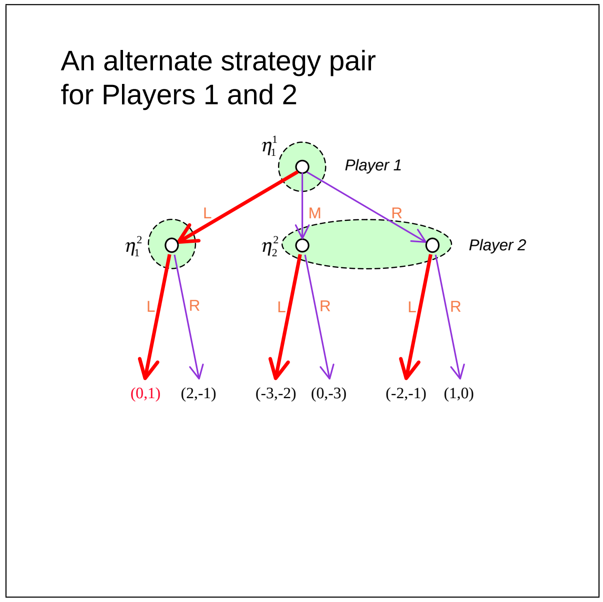
$$\gamma^1(\eta_1^1) = R$$

$$\gamma^2(\eta^2) = \begin{cases} L & \text{if } \eta^2 = \eta_1^2 \\ R & \text{if } \eta^2 = \eta_2^2 \end{cases}$$

The players' task is to choose the best strategy from those available. Using the notation from Section 3.1.1, the set $M_i = \{1, \ldots, m_i\}$ now represents the indices of the possible strategies, $\{\gamma_1^i, \ldots, \gamma_{m_i}^i\}$, for Player i.

Notice that if either player attempts to change his strategy unilaterally, he will not improve his payoff. The above strategy is, in fact, a Nash equilibrium strategy as we will formally define in the next section.

There is another Nash equilibrium strategy for this game, namely



$$\gamma^1(\eta_1^1) = L$$

$$\gamma^2(\eta^2) = \begin{cases} L & \text{if } \eta^2 = \eta_1^2 \\ L & \text{if } \eta^2 = \eta_2^2 \end{cases}$$

But this strategy did not arise from the recursive procedure described in Section 3.2.1. But (γ_1^1, γ_1^2) is, indeed, a Nash equilibrium. Neither player can improve his payoff by a unilateral change in strategy. Oddly, there is no reason for Player 1 to implement this strategy. If Player 1 chooses to go Left, he can only receive 0. But if Player 1 goes Right, Player 2 will go Right, not Left, and Player 1 will receive a payoff of 1. This example shows that games in extensive form can have Nash equilibria that will never be considered for implementation,

3.2.2 Basic ideas

Definition 3.4. *An N-player* **game in extensive form** *is a directed graph with*

1. *a specific vertex indicating the starting point of the game.*
2. *N cost functions each assigning a real number to each terminating node of the graph. The i^{th} cost function represents the gain to Player i if that node is reached.*

3. *a partition of the nodes among the N players.*

4. *a sub-partition of the nodes assigned to Player i into information sets $\{\eta_k^i\}$. The number of branches emanating from each node of a given information set is the same, and no node follows another node in the same information set.*

We will use the following notation:

η^i information sets for Player i.

u^i actual actions for Player i emanating from information sets.

$\gamma^i(\cdot)$ a function whose domain is the set of all information sets $\{\eta^i\}$ and whose range is the set of all possible actions $\{u^i\}$.

The set of $\gamma^i(\cdot)$ is the collection of possible (pure) strategies that Player i could use. In the parlance of economic decision theory, the γ^i are *decision rules.* In game theory, we call them (pure) *strategies.*

For the game illustrated in Section 3.2.1, we can write down all possible strategy pairs (γ^1, γ^2). The text calls these *profiles.*

Player 1 has 3 possible pure strategies:

$$\begin{aligned}\gamma_1^1(\eta_1^1) &= L\\ \gamma_2^1(\eta_1^1) &= M\\ \gamma_3^1(\eta_1^1) &= R\end{aligned}$$

Player 2 has 4 possible pure strategies which can be listed in tabular form, as follows:

	γ_1^2	γ_2^2	γ_3^2	γ_4^2
η_1^2 :	L	R	L	R
η_2^2 :	L	L	R	R

Each strategy pair (γ^1, γ^2), when implemented, results in payoffs to both players which we will denote by $(J^1(\gamma^1, \gamma^2), J^2(\gamma^1, \gamma^2))$. These payoffs produce a game in strategic (normal) form where the rows and columns correspond to the possible pure strategies of Player 1 and Player 2, respectively.

	γ_1^2	γ_2^2	γ_3^2	γ_4^2
γ_1^1	(0,1)	(2,-1)	(0,1)	(2,-1)
γ_2^1	(-3,-2)	(-3,-2)	(0,-3)	(0,-3)
γ_3^1	(-2,-1)	(-2,-1)	(1,0)	(1,0)

Using Definition 3.1, we have two Nash equilibria, namely

$$\begin{array}{lcl}(\gamma_1^1, \gamma_1^2) & \text{with} & J(\gamma_1^1, \gamma_1^2) = (0, 1) \\ (\gamma_3^1, \gamma_3^2) & \text{with} & J(\gamma_3^1, \gamma_3^2) = (1, 0)\end{array}$$

This formulation allows us to

- focus on identifying "good" decision rules even for complicated strategies
- analyze games with different information structures
- analyze multistage games with players taking more than one "turn"

3.2.3 The structure of extensive games

The general definition of games in extensive form can produce a variety of different types of games. This section will discuss some of the approaches to classifying such games. These classification schemes are based on

1. the topology of the directed graph
2. the information structure of the games
3. the sequencing of the players

This section borrows heavily from Başar and Olsder [1]. We will categorize multi-stage games, that is, games where the players take multiple turns. This classification scheme extends to differential games that are played in continuous time. In this section, however, we will use it to classify multi-stage games in extensive form.

Define the following terms:

$\tilde{\eta}_k^i$ information available to Player i at stage k.

x^k state of the game at stage k. This completely describes the current status of the game at any point in time.

$y_k^i = h_k^i(x_k)$ is the state measurement equation, where

$h_k^i(\cdot)$ is the state measurement function

y_k^i is the observation of Player i at state k.

u_k^i decision of Player i at stage k.

The purpose of the function h_k^i is to recognize that the players may not perfect information regarding the current state of the game. The information available to Player i at stage k is then

$$\tilde{\eta}_k^i = \{y_1^1, \ldots, y_k^1; y_1^2, \ldots, y_k^2; \cdots; y_1^N, \ldots, y_k^N\}$$

Based on these ideas, games can be classified as

open loop

$$\tilde{\eta}_k^i = \{x_1\} \quad \forall\, k \in K$$

closed loop, perfect state

$$\tilde{\eta}_k^i = \{x_1, \ldots, x_k\} \quad \forall\, k \in K$$

closed loop, imperfect state

$$\tilde{\eta}_k^i = \{y_1^i, \ldots, y_k^i\} \quad \forall\, k \in K$$

memoryless, perfect state

$$\tilde{\eta}_k^i = \{x_1, x_k\} \quad \forall\, k \in K$$

feedback, perfect state

$$\tilde{\eta}_k^i = \{x_k\} \quad \forall\, k \in K$$

feedback, imperfect state

$$\tilde{\eta}_k^i = \{y_k^i\} \quad \forall\, k \in K$$

Example 3.1. **Princess and the Monster.** This game is played in complete darkness. A princess and a monster know their starting positions in a cave. The game ends when they bump into each other. Princess is trying to maximize the time to the final encounter. The monster is trying to minimize the time. (Open Loop)

Example 3.2. **Lady in the Lake.** This game is played using a circular lake. The lady is swimming with maximum speed v_ℓ. A man (who can't swim) runs along the shore of the lake at a maximum speed of v_m. The lady wins if she reaches shore and the man is not there. (Feedback)

3.3 Structure in extensive form games

I am grateful to Pengfei Yi and Yong Bao who both contributed to Section 3.3.

The solution of an arbitrary extensive form game may require enumeration. But under some conditions, the structure of some games will permit a recursive solution procedure. Many of these results can be found in Başar and Olsder [1].

Definition 3.5. *Player i is said to be a* **predecessor** *of Player j if Player i is closer to the initial vertex of the game's tree than Player j.*

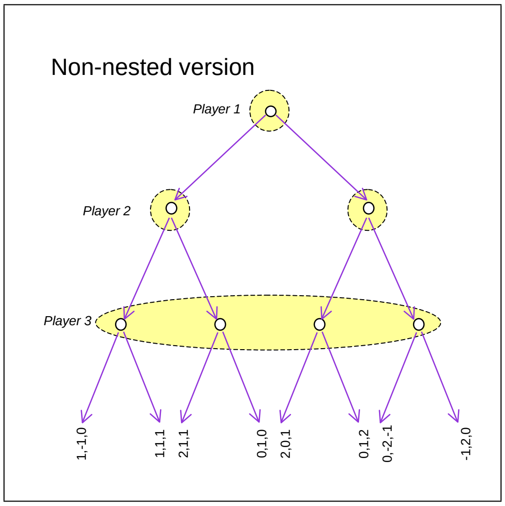
Definition 3.6. *An extensive form game is* **nested** *if each player has access to the information of his predecessors.*

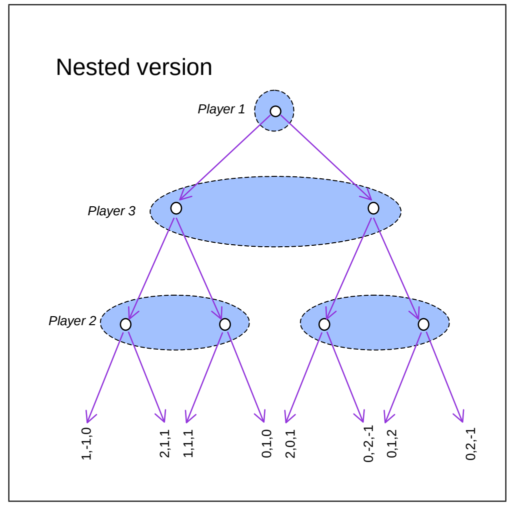
Definition 3.7. (Başar and Olsder [1]) *A nested extensive form game is* **ladder-nested** *if the only difference between the information available to any player (say Player i) and his immediate predecessor (Player $(i-1)$) involves only the actions of Player $(i-1)$, and only at those nodes corresponding to the branches emanating from singleton information sets of Player $(i-1)$.*

Note 3.2. Every 2-player nested game is ladder-nested

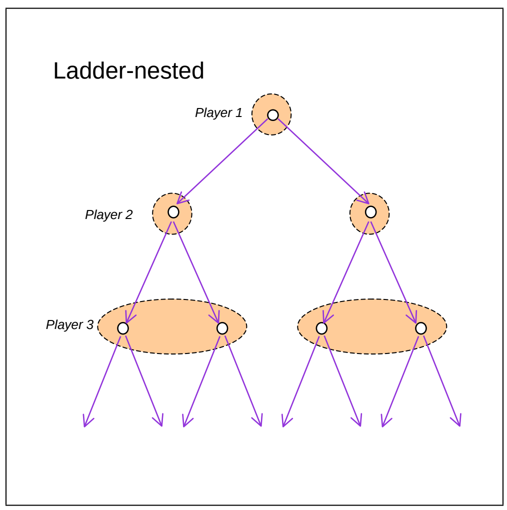
The following three figures illustrate the distinguishing characteristics among non-nested, nested, and ladder-nested games.

The first two figures represent the same single-act game. The first extensive form representation is not nested. The second figure is an extensive form version of the same game that is nested. Thus, we say that this single-act game admits a nested extensive form version.



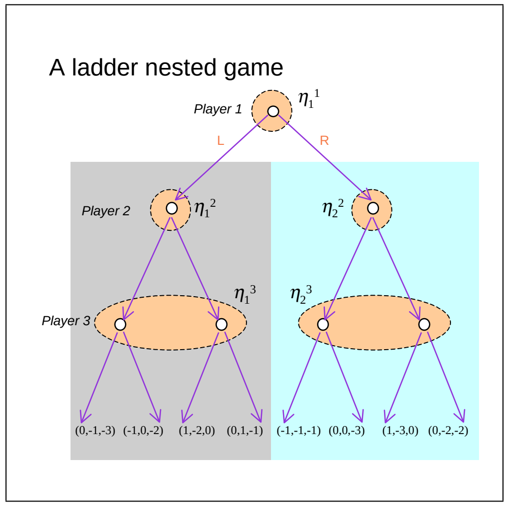


The following figure is an example of a ladder nested game in extensive form.



The important feature of ladder-nested games is that the tree can be decomposed in to sub-trees using the singleton information sets as the starting vertices of the sub-trees. Each sub-tree can then be analyzed as game in strategic form among those players involved in the sub-tree.

As an example, consider the following ladder-nested game:



This game can be decomposed into two bimatrix games involving Player 2 and Player 3. The action of Player 1 determines which of the two games between Player 2 and Player 3 are actually played.

If Player 1 chooses $u^1 = \textbf{L}$ then Player 2 and Player 3 play the game

		Player 3	
		L	**R**
Player 2	**L**	$(-1, -3)$	$(0, -2)$
	R	$(-2, 0)$	$(1, -1)$

Suppose Player 2 uses a mixed strategy of choosing **L** with probability 0.5 and **R** with probability 0.5. Suppose Player 3 also uses a mixed strategy of choosing **L** with probability 0.5 and **R** with probability 0.5. Then these mixed strategies are a Nash equilibrium solution for this sub-game with an expected payoff to all three players of $(0, -0.5, -1.5)$.

If Player 1 chooses $u^1 = \textbf{R}$ then Player 2 and Player 3 play the game

		Player 3	
		L	**R**
Player 2	**L**	$(-1, -1)$	$(0, -3)$
	R	$(-3, 0)$	$(0, -2)$

This subgame has a Nash equilibrium in pure strategies with Player 2 and Player 3 both choosing **L**. The payoff to all three players in this case is of $(-1, -1, -1)$.

To summarize the solution for all three players we will introduce the concept of a *behavioral strategy:*

Definition 3.8. *A* **behavioral strategy** *(or* **locally randomized strategy***) assigns for each information set a probability vector to the alternatives emanating from the information set.*

When using a behavioral strategy, a player simply randomizes over the alternatives from each information set. When using a mixed strategy, a player randomizes his selection from the possible pure strategies for the entire game.

The following behavioral strategy produces a Nash equilibrium for all three players:

$$
\begin{aligned}
\gamma^1(\eta_1^1) &= \mathsf{L} \\
\gamma^2(\eta_1^2) &= \begin{cases} \mathsf{L} & \text{with probability } 0.5 \\ \mathsf{R} & \text{with probability } 0.5 \end{cases} \\
\gamma^2(\eta_2^2) &= \begin{cases} \mathsf{L} & \text{with probability } 1 \\ \mathsf{R} & \text{with probability } 0 \end{cases} \\
\gamma^3(\eta_1^3) &= \begin{cases} \mathsf{L} & \text{with probability } 0.5 \\ \mathsf{R} & \text{with probability } 0.5 \end{cases} \\
\gamma^3(\eta_2^3) &= \begin{cases} \mathsf{L} & \text{with probability } 1 \\ \mathsf{R} & \text{with probability } 0 \end{cases}
\end{aligned}
$$

with an expected payoff of $(0, -0.5, -1.5)$.

Note 3.3. When using a behavioral strategy, a player, at each information set, must specify a probability distribution over the alternatives for that information set. It is assumed that the choices of alternatives at different information sets are made independently. Thus it might be reasonable to call such strategies "uncorrelated" strategies.

Note 3.4. For an arbitrary game, not all mixed strategies can be represented by using behavioral strategies. Behavioral strategies are easy to find and represent. We would like to know when we can use behavioral strategies instead of enumerating all pure strategies and randomizing among those pure strategies.

Theorem 3.3. *Every single-stage, ladder-nested N-person game has at least one Nash equilibrium using behavioral strategies.*

3.3.1 An example by Kuhn

One can show that every behavioral strategy can be represented as a mixed strategy. But an important question arises when considering mixed strategies vis-à-vis behavioral strategies: Can a mixed strategy always be represented by a behavioral strategy?

The following example from Kuhn [2] shows a remarkable result involving behavioral strategies. It shows what can happen if the players do not have a property called *perfect recall*. Moreover, the property of *perfect recall* alone is a necessary and sufficient condition to obtain a one-to-one mapping between behavioral and mixed strategies for any game.

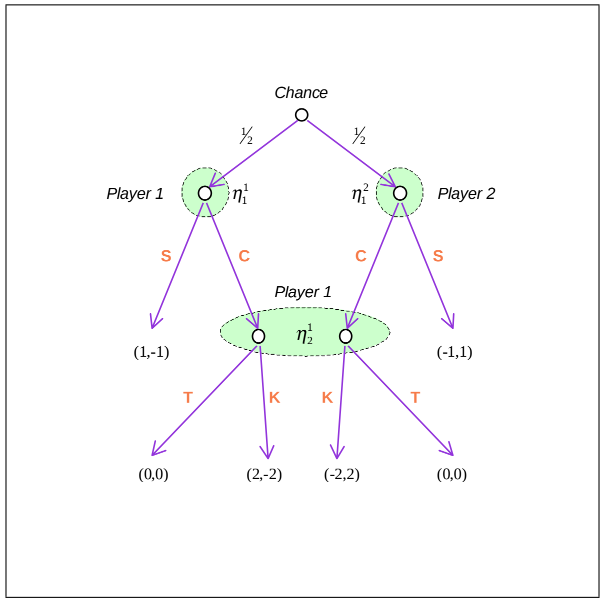
In a game with **perfect recall,** each player remembers everything he knew at previous moves and all of his choices at these moves.

A zero-sum game involves two players and a deck of cards. A card is dealt to each player. If the cards are not different, two more cards are dealt until one player has a higher card than the other.

The holder of the high card receives $1 from his opponent. The player with the high card can choose to either stop the game or continue.

If the game continues, Player 1 (who forgets whether he has the high or low card) can choose to leave the cards as they are or trade with his opponent. Another $1 is then won by the (possibly different) holder of the high card.

The game can be represented with the following diagram:



where

S	Stop the game
C	Continue the game
T	Trade cards
K	Keep cards

At information set η_1^1, Player 1 makes the critical decision that causes him to eventually lose perfect recall at η_2^1. Moreover, it is Player 1's own action that causes this loss of information (as opposed to Player 2 causing the loss). This is the reason why behavioral strategies fail for Player 1 in this problem.

Define the following pure strategies for Player 1:

$$\gamma_1^1(\eta^1) = \begin{cases} \mathbf{S} & \text{if } \eta^1 = \eta_1^1 \\ \mathbf{T} & \text{if } \eta^1 = \eta_2^1 \end{cases} \qquad \gamma_2^1(\eta^1) = \begin{cases} \mathbf{S} & \text{if } \eta^1 = \eta_1^1 \\ \mathbf{K} & \text{if } \eta^1 = \eta_2^1 \end{cases}$$

$$\gamma_3^1(\eta^1) = \begin{cases} \mathbf{C} & \text{if } \eta^1 = \eta_1^1 \\ \mathbf{T} & \text{if } \eta^1 = \eta_2^1 \end{cases} \qquad \gamma_4^1(\eta^1) = \begin{cases} \mathbf{C} & \text{if } \eta^1 = \eta_1^1 \\ \mathbf{K} & \text{if } \eta^1 = \eta_2^1 \end{cases}$$

and for Player 2:

$$\gamma_1^2(\eta_1^2) = \mathbf{C} \qquad \gamma_2^2(\eta_1^2) = \mathbf{S}$$

This results in the following strategic (normal) form game:

	γ_1^2	γ_2^2
γ_1^1	$(1/2, -1/2)$	$(0, 0)$
γ_2^1	$(-1/2, 1/2)$	$(0, 0)$
γ_3^1	$(0, 0)$	$(-1/2, 1/2)$
γ_4^1	$(0, 0)$	$(1/2, -1/2)$

Question 3.2. Show that the mixed strategy for Player 1:

$$(\tfrac{1}{2}, 0, 0, \tfrac{1}{2})$$

and the mixed strategy for Player 2:

$$(\tfrac{1}{2}, \tfrac{1}{2})$$

result in a Nash equilibrium with expected payoff $(\frac{1}{4}, -\frac{1}{4})$.

Question 3.3. Suppose that Player 1 uses a behavioral strategy (x, y) defined as follows: Let $x \in [0, 1]$ be the probability Player 1 chooses **S** when he is in information set η_1^1, and let $y \in [0, 1]$ be the probability Player 1 chooses **T** when he is in information set η_2^1.

Also suppose that Player 2 uses a behavioral strategy (z) where $z \in [0, 1]$ is the probability Player 2 chooses **S** when he is in information set η_1^2.

Let $E^i((x, y), z)$ denote the expected payoff to Player $i = 1, 2$ when using behavioral strategies (x, y) and (z). Show that,

$$E^1((x, y), z) = (x - z)(y - \tfrac{1}{2})$$

and $E^1((x, y), z) = -E^2((x, y), z)$ for any x, y and z.

Furthermore, consider

$$\max_{x,y} \min_{z} (x - z)(y - \tfrac{1}{2})$$

and show that the every equilibrium solution in behavioral strategies must have $y = \frac{1}{2}$ where

$$E^1((x, \tfrac{1}{2}), z) = -E^2((x, \tfrac{1}{2}), z) = 0.$$

Therefore, using only behavioral strategies, the expected payoff will be $(0, 0)$. If Player 1 is restricted to using only behavioral strategies, he can guarantee, at most, an expected gain of 0. But if he randomizes over all of his pure strategies and stays with that strategy throughout the game, Player 1 can get an expected payoff of $\frac{1}{4}$.

Any behavioral strategy can be expressed as a mixed strategy. But, without perfect recall, not all mixed strategies can be implemented using behavioral strategies.

Theorem 3.4. (Kuhn [2]) *Perfect recall is a necessary and sufficient condition for all mixed strategies to be induced by behavioral strategies.*

A formal proof of this theorem is in [2]. Here is a brief sketch: We would like to know under what circumstances there is a 1-1 correspondence between behavioral and mixed strategies. Suppose a mixed strategy consists of the following mixture of three pure strategies:

$$\begin{array}{lll} \text{choose} & \gamma_a & \text{with probability } \frac{1}{2} \\ \text{choose} & \gamma_b & \text{with probability } \frac{1}{3} \\ \text{choose} & \gamma_c & \text{with probability } \frac{1}{6} \end{array}$$

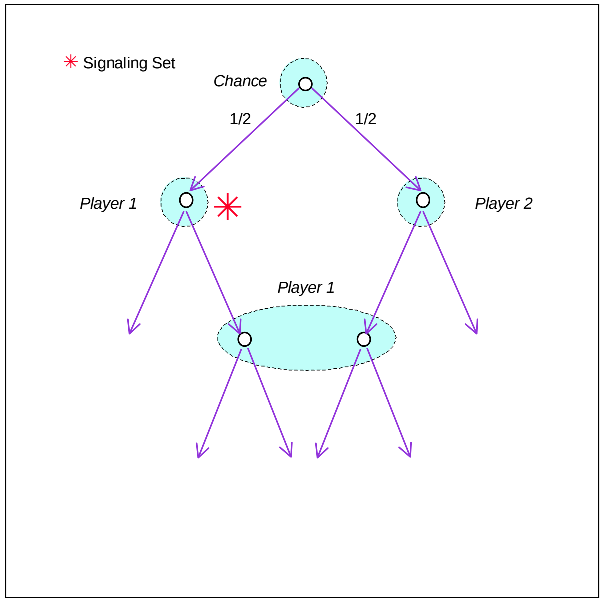
Suppose that strategies γ_b and γ_c lead the game to information set η. Suppose that strategy γ_a does not go to η. If a player is told he is in information η, he can use perfect recall to backtrack completely through the game to learn whether strategy γ_b or γ_c was used. Suppose $\gamma_b(\eta) = u_b$ and $\gamma_c(\eta) = u_c$. Then if the player is in η, he can implement the mixed strategy with the following behavioral strategy:

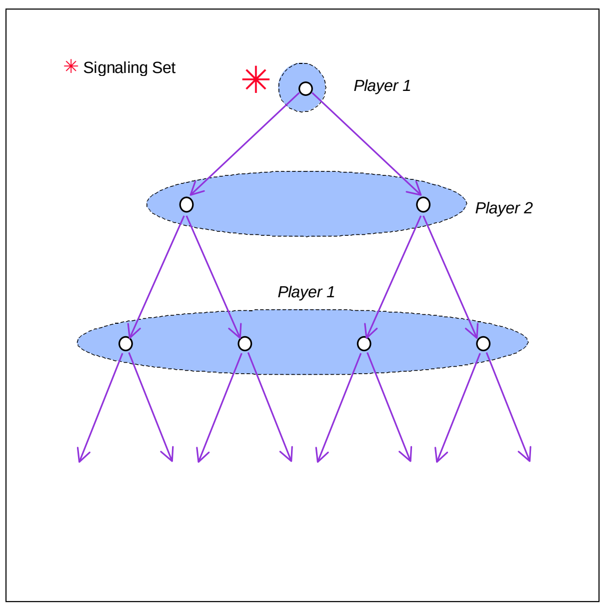
$$\begin{array}{lll} \text{choose} & u_b & \text{with probability } \frac{2}{3} \\ \text{choose} & u_c & \text{with probability } \frac{1}{3} \end{array}$$

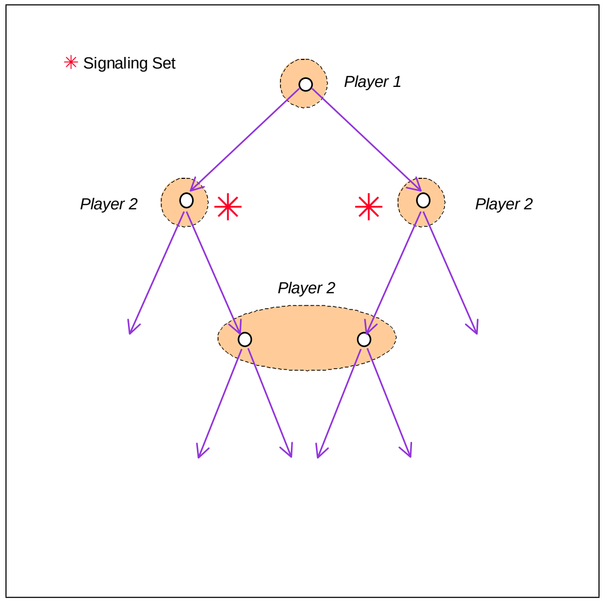
3.3.2 Signaling information sets

A game may not have perfect recall, but some strategies could take the game along paths that, as sub-trees, have the property of perfect recall. Kuhn [2] and Thompson [4] employ the concept of **signaling information sets**. In essence, a signaling information set is that point in the game where a decision by a player could cause him to lose the property of perfect recall.

In the following three games, the signaling information sets are marked with (*****):







3.4 Stackelberg solutions

3.4.1 Basic ideas

This early idea in game theory is due to Stackelberg [3]. Its features include:

- hierarchical ordering of the players
- strategy decisions are made and announced sequentially
- one player has the ability to enforce his strategy on others

This approach introduce is notion of a *rational reaction* of one player to another's choice of strategy.

Example 3.3. Consider the bimatrix game

	γ_1^2	γ_2^2	γ_3^2
γ_1^1	$(0,1)$	$(-2,-1)$	$(-\frac{3}{2},-\frac{2}{3})$
γ_2^1	$(-1,-2)$	$(-1,0)$	$(-3,-1)$
γ_2^1	$(1,0)$	$(-2,-1)$	$(-2,\frac{1}{2})$

Note that (γ_2^1, γ_2^2) is a Nash solution with value $(-1, 0)$.

Suppose that Player 1 must "lead" by announcing his strategy, first. Is this an advantage or disadvantage? Note that,

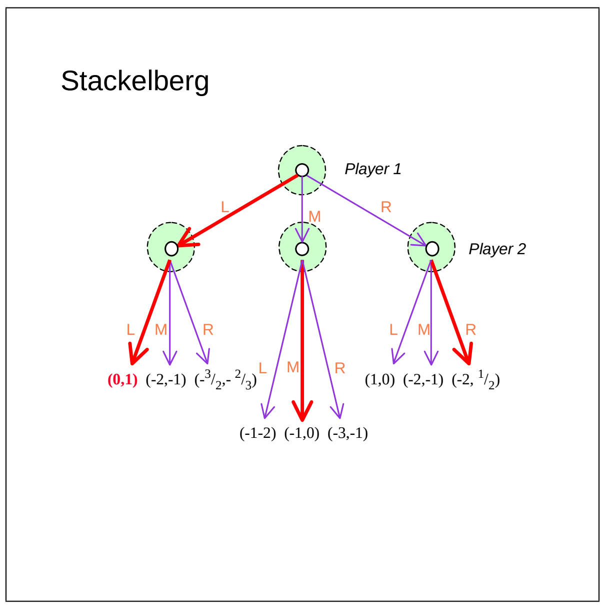
If Player 1 chooses	γ_1^1	Player 2 will respond with	γ_1^2
If Player 1 chooses	γ_2^1	Player 2 will respond with	γ_2^2
If Player 1 chooses	γ_3^1	Player 2 will respond with	γ_3^2

The best choice for Player 1 is γ_1^1 which will yield a value of $(0, 1)$. For this game, the **Stackelberg solution** is an improvement over the Nash solution for *both players.*

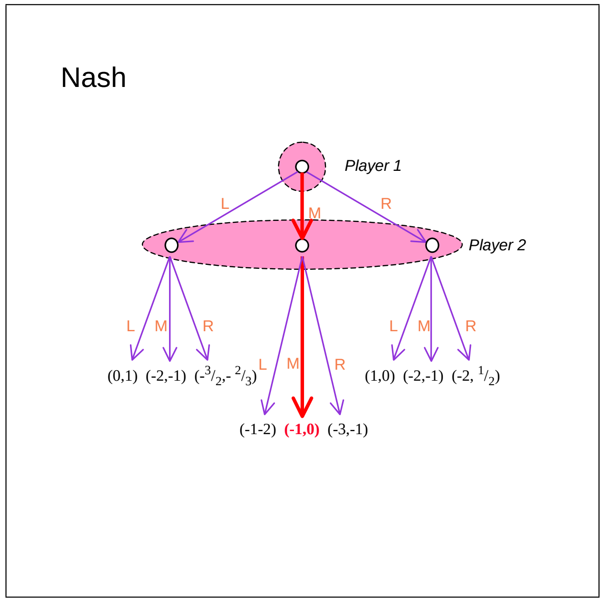
If we let

$$\begin{aligned} \gamma_1^1 &= \textbf{L} & \qquad \gamma_1^2 &= \textbf{L} \\ \gamma_2^1 &= \textbf{M} & \qquad \gamma_2^2 &= \textbf{M} \\ \gamma_3^1 &= \textbf{R} & \qquad \gamma_3^2 &= \textbf{R} \end{aligned}$$

we can implement the Stackelberg strategy by playing the following game in extensive form:



The Nash solution can be obtained by playing the following game:



There may not be a unique response to the leader's strategy. Consider the following example:

	γ_1^2	γ_2^2	γ_3^2
γ_1^1	$(0,0)$	$(-1,0)$	$(-3,-1)$
γ_2^1	$(-2,1)$	$(-2,0)$	$(1,1)$

In this case

If Player 1 chooses	γ_1^1	Player 2 will respond with	γ_1^2 or γ_2^2
If Player 1 chooses	γ_2^1	Player 2 will respond with	γ_1^2 or γ_3^2

One solution approach uses a minimax philosophy. That is, Player 1 should secure his profits against the alternative rational reactions of Player 2. If Player 1 chooses γ_1^1 the least he will obtain is -1, and he chooses γ_2^1 the least he will obtain is -2. So his (minimax) Stackelberg strategy is γ_1^1.

Question 3.4. In this situation, one might consider mixed Stackelberg strategies. How could such strategies be defined, when would they be useful, and how would they be implemented?

Note 3.5. When the follower's response is not unique, a natural solution approach would be to *side-payments*. In other words, Player 1 could provide an incentive to Player 2 to choose an action in Player 1's best interest. Let $\epsilon > 0$ be a small side-payment. Then the players would be playing the Stackelberg game

	γ_1^2	γ_2^2	γ_3^2
γ_1^1	$(-\epsilon, \epsilon)$	$(-1, 0)$	$(-3, -1)$
γ_2^1	$(-2, 1)$	$(-2, 0)$	$(1 - \epsilon, 1 + \epsilon)$

3.4.2 The formalities

Let Γ^1 and Γ^2 denote the sets of pure strategies for Player 1 and Player 2, respectively. Let $J^i(\gamma^1, \gamma^2)$ denote the payoff to Player i if Player 1 chooses strategy $\gamma^1 \in \Gamma^1$ and Player 2 chooses strategy $\gamma^2 \in \Gamma^2$. Let

$$R^2(\gamma^1) \equiv \{\xi \in \Gamma^2 \mid J^2(\gamma^1, \xi) \geq J^2(\gamma^1, \gamma^2) \; \forall \, \gamma^2 \in \Gamma^2\}$$

Note that $R^2(\gamma^1) \subseteq \Gamma^2$ and we call $R^2(\gamma^1)$ the **rational reaction** of Player 2 to Player 1's choice of γ^1. A **Stackelberg strategy** can be formally defined as the $\hat{\gamma}^1$ that solves

$$\min_{\gamma^2 \in R^2(\hat{\gamma}^1)} J^1(\hat{\gamma}^1, \gamma^2) = \max_{\gamma^1 \in \Gamma^1} \min_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2) = J^{1*}$$

Note 3.6. If $R^2(\gamma^1)$ is a singleton for all $\gamma^1 \in \Gamma^1$ then there exists a mapping

$$\psi^2 : \Gamma^1 \to \Gamma^2$$

such that $R^2(\gamma^1) = \{\gamma^2\}$ implies $\gamma^2 = \psi^2(\gamma^1)$. In this case, the definition of a Stackelberg solution can be simplified to the $\hat{\gamma}^1$ that solves

$$J^1(\hat{\gamma}^1, \psi^2(\gamma^1)) = \max_{\gamma^1 \in \Gamma^1} J^1(\gamma^1, \psi^2(\gamma^1))$$

It is easy to prove the following:

Theorem 3.5. *Every two-person finite game has a Stackelberg solution for the leader.*

Note 3.7. From the follower's point of view, his choice of strategy in a Stackelberg game is always optimal (i.e., the best he can do).

Question 3.5. Let J^{1*} (defined as above) denote the Stackelberg value for the leader Player 1, and let J^1_N denote any Nash equilibrium solution value for the same player. What is the relationship (bigger, smaller, etc.) between J^{1*} and J^1_N? What additional conditions (if any) do you need to place on the game to guarantee that relationship?

3.5 BIBLIOGRAPHY

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